

VALIDATED EXPONENTIAL ANALYSIS FOR HARMONIC SOUNDS

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ABSTRACT

In audio spectral analysis, the Fourier method is popular because of its stability and its low computational complexity. It suffers however from a time-frequency resolution trade off and is not particularly suited for aperiodic signals such as exponentially decaying ones. To overcome their resolution limitation, additional techniques such as quadratic peak interpolation or peak picking, and instantaneous frequency computation from phase unwrapping are used.

Parametric methods on the other hand, overcome the time-frequency trade off but are more susceptible to noise and have a higher computational complexity. We propose a method to overcome these drawbacks: we set up regularized smaller sized independent problems and perform a cluster analysis on their combined output. The new approach validates the true physical terms in the exponential model, is robust in the presence of outliers in the data and is able to filter out any non-physical noise terms in the model.

The method is illustrated in the removal of electrical humming in harmonic sounds.

1. INTRODUCTION

Multi-exponential models arise naturally in music and audio. They are used in audio effects, audio coding, source separation and music transcription. Multi-exponential models can be computed via a non-parametric or a parametric approach.

The non-parametric Fourier based techniques are widely used to decompose a signal into a sum of complex exponentials with equispaced frequencies on the unit circle. The frequency resolution is directly linked to the length of the time window used for the analysis. Longer time windows result in a higher frequency resolution, thus bounding these methods to quasi-stationary recordings. To overcome this limitation and improve the frequency resolution of short time windows, one can exploit the amplitude or phase information of consecutive windows. An overview of such techniques is found in [1].

On the other hand, parametric methods can overcome the frequency limitation of the Fourier based methods, but at the expense of being more susceptible to the noise present in the signal. Another advantage of parametric methods is that they can model complex exponentials whose amplitudes are modulated by exponential decays or polynomials [2]. These methods are successfully applied in different fields of audio signal processing [3, 4, 5]. To improve the robustness against noise, other approaches have also been developed [6, 7].

Recently multi-exponential analysis was generalized [8] to sub-Nyquist sampling rates. The ability to use a coarser time grid may also improve the numerical stability of the parametric method. Moreover, the use of a uniform non-consecutive sampling scheme permits to avoid outliers in the data. At the same time, connections of the method with sparse interpolation from computer algebra and Padé approximation from numerical approximation theory were studied in depth [9]. In the Sections 2, 3 and 4 all these recent results are recapped.

Making use of these recent developments, we present a new approach in Section 5, which can be combined with the use of existing parametric methods. In particular, it can be used on top of any parametric technique derived from Prony's method. The new parametric analysis is performed on several decimated signals, each being sub-sampled. The use of a cluster detection algorithm allows to automatically validate the output of the method. In the same way it is possible to remove outliers from the data since the latter will not be validated as true frequencies.

When applying the proposed method in Section 6 to harmonic sounds, we see that it easily allows to detect and numerically refine the lowest partial of an harmonic sound, since the validated frequencies retrieved by the cluster analysis are stable frequencies. From the lowest partial, a denoised version of the source can subsequently be constructed.

2. THE MULTI-EXPONENTIAL MODEL IN SIGNAL PROCESSING

In order to proceed we introduce some notations. Let the real numbers $\psi_i, \omega_i, \beta_i$ and γ_i respectively denote the damping, frequency, amplitude and phase in each component of the signal

$$\phi(t) = \sum_{i=1}^n \alpha_i \exp(\phi_i t), \quad t \in \mathbb{Z}$$

$$i^2 = -1 \quad \alpha_i = \beta_i \exp(i\gamma_i), \quad \phi_i = \psi_i + i2\pi\omega_i. \quad (1)$$

For the moment, we assume that the frequency content of $\phi(t)$ is limited by

$$|\Im(\phi_i)/(2\pi)| = |\omega_i| < \Omega/2, \quad i = 1, \dots, n,$$

and we sample $\phi(t)$ at the equidistant points $t_j = j\Delta$ for $j = 0, 1, \dots, 2n-1, \dots, N$ with $\Delta \leq 1/\Omega$. In the sequel we denote

$$f_j := \phi(t_j), \quad j = 0, 1, \dots, 2n-1, \dots, N.$$

The aim is to find the model order n , and the parameters ϕ_1, \dots, ϕ_n and $\alpha_1, \dots, \alpha_n$ from the measurements $f_0, \dots, f_{2n}, \dots$. We further denote

$$\lambda_i := \exp(\phi_i \Delta), \quad i = 1, \dots, n.$$

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With

$$H_n^{(r)} := \begin{pmatrix} f_r & \cdots & f_{r+n-1} \\ \vdots & \ddots & \vdots \\ f_{r+n-1} & \cdots & f_{r+2n-2} \end{pmatrix}, \quad r \geq 0, n \geq 1,$$

the λ_i are retrieved [10] as the generalized eigenvalues of the problem

$$H_n^{(1)} v_i = \lambda_i H_n^{(0)} v_i, \quad i = 1, \dots, n, \quad (2)$$

where v_i are the generalized right eigenvectors. From the values λ_i , the complex numbers ϕ_i can be retrieved uniquely because of the restriction $|\Im(\phi_i \Delta)| < \pi$. In a noisy context the Hankel matrices can be extended to be rectangular of size $m \times n$ with $m > n$ and equation (2) can be considered in a least square sense [11].

In the absence of noise, the exact value for n can be deduced from [12, p. 603] (for a detailed discussion see [13])

$$\begin{aligned} \det H_n^{(r)} &\neq 0, \\ \det H_\nu^{(r)} &= 0, \quad \nu > n. \end{aligned} \quad (3)$$

In the presence of noise and/or clusters of eigenvalues, the use of (3) is not very reliable though. We indicate in Section 5 how n is then to be detected numerically.

Finally, the α_i are computed from the interpolation conditions

$$\sum_{i=1}^n \alpha_i \exp(\phi_i t_j) = f_j, \quad j = 0, \dots, 2n-1, \quad (4)$$

either by solving the system in the least squares sense, in the presence of noise, or by solving a subset of n (consecutive) interpolation conditions in case of a noise-free $\phi(t)$. Note that

$$\exp(\phi_i t_j) = \lambda_i^j$$

and that the coefficient matrix of (4) is therefore a Vandermonde matrix. It is well-known that the conditioning of structured matrices is something that needs to be monitored [14, 15].

3. EXPONENTIAL ANALYSIS VIEWED AS PADÉ APPROXIMATION

With $f_j = \phi(t_j)$ we now define the noise-free

$$f(z) = \sum_{j=0}^{\infty} f_j z^j. \quad (5)$$

The Padé approximant $r_{m,n}(z)$ of degree m in the numerator and n in the denominator is defined as the irreducible form of the rational function $p(z)/q(z)$ satisfying

$$\frac{d^j (fq - p)(z)}{dt^j} (0) = 0, \quad j = 0, \dots, m+n.$$

This condition guarantees a high degree of contact between $f(z)$ and $p(z)/q(z)$. Since

$$f_j = \sum_{i=1}^n \alpha_i \exp(\phi_i j \Delta) = \sum_{i=1}^n \alpha_i \lambda_i^j,$$

we can rewrite

$$f(z) = \sum_{i=1}^n \frac{\alpha_i}{1 - z \lambda_i}. \quad (6)$$

So we see that $f(z)$ is itself a rational function of degree $n-1$ in the numerator and n in the denominator, with poles $1/\lambda_i$. Hence, from Padé approximation theory we know that $r_{n-1,n}(z)$ computed for (5) reconstructs $f(z)$, in other words

$$r_{n-1,n}(z) = f(z).$$

The partial fraction decomposition (6) is related to the Laplace transform of the exponential model (1), which explains why this approach is known as the Padé-Laplace method [16].

Now we add a white circular Gaussian noise term ϵ_j to each sample f_j . In the sequel we denote the noisy series by

$$f(z) + \epsilon(z) = \sum_{j=0}^{\infty} (f_j + \epsilon_j) z^j.$$

A number of strong approximation and convergence results exist for sequences of Padé approximants to $f(z) + \epsilon(z)$. They express what one would expect intuitively from such approximants: they are especially useful if the approximated function is meromorphic (i.e. has poles) in some substantial region of the complex plane [17], as is the case for $f(z)$ given by (6). The theorem of Nuttall, later generalized by Pommerenke, states that if $f(z) + \epsilon(z)$ is analytic throughout the complex plane except for a countable number of poles [18] and essential singularities [19], then the paradiagonal sequence $\{r_{\nu-1,\nu}(z)\}_{\nu \in \mathbb{N}}$ converges to $f(z) + \epsilon(z)$ in measure on compact sets. So no assertion is made about pointwise or uniform convergence. Instead, the result states that for sufficiently large ν , the measure of the set where the convergence is disrupted, so where $|f(z) + \epsilon(z) - r_{\nu-1,\nu}(z)| \geq \tau$ for some given threshold τ , tends to zero as ν tends to infinity. The pointwise convergence is actually disrupted by $\nu - n$ unwanted pole-zero combinations of the Padé approximants $r_{\nu-1,\nu}(z)$: near each spurious pole introduced by increasing the denominator degree beyond the true n , one finds an associated zero, the pole and zero effectively cancelling each other locally. These pole-zero doublets are referred to as Froissart doublets [20, 21, 22]. Because of the Padé convergence theorem, the n true physical poles can be identified as stable poles of successive $r_{\nu-1,\nu}(z)$, while the $\nu - n$ spurious nonphysical poles are distinguished by their instability [23, 24]. So these Froissart doublets offer a way to filter the noise $\epsilon(z)$ from the underlying signal $f(z)$ [25]. Because of their ability to model the noise, Froissart doublets should not be avoided in the computation, as in [26] and [27], but should be filtered out at a later stage in the computation, an approach we apply in this paper.

So n is revealed as the number of stable poles in successive Padé approximants $r_{\nu-1,\nu}(z)$ for (6). Before moving to an implementation of this idea, we need another tool.

4. REGULARIZATION OF THE PROBLEM BY DOWNSAMPLING

When replacing Δ by a multiple

$$\Delta(k) := k\Delta, \quad \Delta(k) \leq k/\Omega$$

and thus sampling at $t_{jk} = j\Delta(k) = jk\Delta$, the eigenvalues we retrieve from (2) are not λ_i , but

$$\lambda_i(k) = \lambda_i^k, \quad i = 1, \dots, n.$$

So we fill the Hankel matrices $H_n^{(r)}$ with the samples f_{ik} instead of the samples f_i , $i = 0, \dots, 2n - 1$. To avoid confusion we denote the latter ones by

$$H_n^{(r)}(k) := \begin{pmatrix} f_{rk} & \dots & f_{(r+n-1)k} \\ \vdots & \ddots & \vdots \\ f_{(r+n-1)k} & \dots & f_{(r+2n-2)k} \end{pmatrix}.$$

Again these Hankel matrices can be extended to be rectangular of size $m \times n$ with $m > n$. From $\lambda_i^k = \exp(k\phi_i\Delta)$ the imaginary part of ϕ_i cannot be retrieved uniquely anymore, because now

$$|\Im(k\phi_i\Delta)| < k\pi.$$

So aliasing may have occurred: because of the periodicity of the function $\exp(i2\pi k\omega_i\Delta)$ a total of k values in the $2k\pi$ wide interval can be identified as plausible values for $2\pi\omega_i$. Note that when the original λ_i are clustered, the powered λ_i^k may be distributed quite differently and unclustered. Such a relocation of the generalized eigenvalues may significantly improve the conditioning of the Hankel matrices involved.

What remains is to investigate how to solve the aliasing problem in the imaginary parts $2\pi\omega_i$ of the ϕ_i . So far, from λ_i^k , we only have aliased values for ω_i . But this aliasing can be fixed at the expense of a small number of additional samples. In what follows n can everywhere be replaced by $\nu > n$ when using $\nu - n$ additional terms to model the noise, and all square $\nu \times \nu$ matrices can be replaced by rectangular $\mu \times \nu$ counterparts with $\mu > \nu$. To fix the aliasing, we add n samples to the set $\{f_0, f_k, \dots, f_{(2n-1)k}\}$, namely at the shifted points

$$t_{kj+\kappa} = jk\Delta + \kappa\Delta = jk\Delta + \kappa\Delta, \\ j = r, \dots, r + n - 1, \quad 0 \leq r \leq n.$$

An easy choice for κ is a small number relatively prime with k (for the most general choice allowed, we refer to [28]). With the additional samples we proceed as follows.

From the samples $\{f_0, f_k, \dots, f_{(2n-1)k}, \dots\}$ we compute the generalized eigenvalues $\lambda_i^k = \exp(\phi_i k\Delta)$ and the coefficients α_i going with λ_i^k in the model

$$\begin{aligned} \phi(jk\Delta) &= \sum_{i=1}^n \alpha_i \exp(\phi_i jk\Delta) \\ &= \sum_{i=1}^n \alpha_i \lambda_i^{jk}, \quad j = 0, \dots, 2n - 1. \end{aligned} \quad (7)$$

So we know which coefficient α_i goes with which generalized eigenvalue λ_i^k , but we just cannot identify the correct $\Im(\phi_i)$ from λ_i^k . The samples at the additional points $t_{jk+\kappa}$ satisfy

$$\begin{aligned} \phi(jk\Delta + \kappa\Delta) &= \sum_{i=1}^n \alpha_i \exp(\phi_i(jk + \kappa)\Delta) \\ &= \sum_{i=1}^n (\alpha_i \lambda_i^\kappa) \lambda_i^{jk}, \quad j = r, \dots, r + n - 1, \end{aligned} \quad (8)$$

which can be interpreted as a linear system with the same coefficients matrix as (7), but now with a new left hand side and unknowns $\alpha_1 \lambda_1^\kappa, \dots, \alpha_n \lambda_n^\kappa$ instead of $\alpha_1, \dots, \alpha_n$. And again we

can associate each computed $\alpha_i \lambda_i^\kappa$ with the proper generalized eigenvalue λ_i^k . Then by dividing the $\alpha_i \lambda_i^\kappa$ computed from (8) by the α_i computed from (7), for $i = 1, \dots, n$, we obtain from λ_i^κ a second set of κ plausible values for ω_i . Because of the fact that we choose κ and k relatively prime, the two sets of plausible values for ω_i have only one value in their intersection [29]. Thus the aliasing problem is solved.

5. ADDING VALIDATION TO THE ANALYSIS METHOD

The Padé view from Section 3 can now nicely be combined with the regularization technique from Section 4. The downsampling option uses only $1/k$ of the overall sequence of samples $f_0, f_1, \dots, f_{2\nu-1}, \dots$ taken at equidistant points $t_j = j\Delta$, $j = 0, 1, 2, \dots$, plus an additional set of samples at shifted locations to get rid of some possible aliasing effect. So for a fixed $k > 0$ the full sequence of samples points can be divided into k downsampled sub-sequences

$$\begin{aligned} &t_0, t_k, t_{2k}, \dots \\ &t_1, t_{k+1}, t_{2k+1}, \dots \\ &\vdots \\ &t_{k-1}, t_{2k-1}, t_{3k-1}, \dots \end{aligned}$$

For each downsampled subsequence $t_{\ell+kj}$, $\ell = 0, \dots, k - 1$ a sequence of shifted sample points $t_{\ell+kj+\kappa}$ can also be extracted from the original full sequence t_j , as long as $\gcd(k, \kappa) = 1$. Actually, the computation of λ_i^κ can be improved numerically by considering the following shift strategy instead of a single shift. After the first shift by κ of the sample points to $t_{\ell+kj+\kappa}$, multiples of the shift κ can be considered. By sampling at $t_{\ell+kj+h\kappa}$, $h = 1, \dots, H = \lfloor (N - k - 1 - 2n\kappa)/\kappa \rfloor$ the values $\alpha_i \lambda_i^\kappa$, $\alpha_i \lambda_i^{2\kappa}$, \dots , $\alpha_i \lambda_i^{H\kappa}$ are obtained, which can be considered as the samples $\psi(1), \psi(2), \dots, \psi(H)$ of a mono-exponential analysis problem

$$\psi(h) = \alpha_i \lambda_i^{\kappa h}. \quad (9)$$

From these we can compute λ_i^κ using (2) for a single exponential term. For this the Hankel matrices are again best enlarged to rectangular matrices. This sparse interpolation replaces the division $\alpha_i \lambda_i^\kappa / \alpha_i$ by a more accurate procedure.

By repeating the computation of λ_i^k from (7) and λ_i^κ from (9) for each $\ell = 0, \dots, k - 1$, we obtain k independent problems of the form (8) instead of just one. In each of these – assuming that we overshoot the true number of components n in (7) and (8) by considering $\nu - n - \kappa$ the true parameters ϕ_i from the model (1) appear as n stable poles in the Padé-Laplace method and the $\nu - n$ spurious noisy poles behave in an unstable way. In fact, each downsampled sequence can be seen as a different noise realization while the underlying function $\phi(t)$ remains the same. So the generalized eigenvalues related to the signal $\phi(t)$ cluster near the true λ_i^k , while the other generalized eigenvalues belong to independent noise realizations and do not form clusters anywhere [23, 24].

The gain of considering k downsampled multi-exponential analysis problems rather than one large problem is twofold:

- A cluster analysis algorithm can detect the number of clusters in the complex plane, and hence deliver the number n of components in the model of the form (1). So there is no need to estimate the number n separately, by means of an SVD of $H_\nu^{(0)}$ with $\nu > n$ for instance.

- Because the physically meaningful generalized eigenvalues form clusters of (about) k elements, their value can be estimated more accurately by computing the center of gravity of each cluster, where the cluster radius is a function of the ill-disposedness of that specific λ_i [15].

The cluster analysis method used in the examples below is DBSCAN [30]. Since the cluster radii may vary, we typically perform two runs of DBSCAN with different parameter settings. In a first run we retrieve the clusters with higher density, while a second run allows to detect the less dense clusters of generalized eigenvalues.

After obtaining a center of gravity as approximation for the λ_i^k and a center of gravity as approximation for the λ_i^κ associated to the λ_i^k , the intersection of the solution sets for ω_i can be taken. We simply build a distance matrix and look for the closest match between

$$\omega_i : \exp(k\phi_i\Delta) = \exp(k(\psi_i + i2\pi\omega_i)\Delta) = \lambda_i^k$$

and

$$\omega_i : \exp(\kappa\phi_i\Delta) = \exp(\kappa(\psi_i + i2\pi\omega_i)\Delta) = \lambda_i^\kappa.$$

We point out and emphasize that the above technique can be combined with any implementation to solve problem (1), more precisely (7) and (8), popular methods being [31, 10, 11].

To illustrate the validation aspect and how it is robust in the presence of outliers, we consider 400 audio samples of the sustained part of an A4 note played by a trumpet, corrupted by an outlier as shown in Figure 1. We put $k = 4$ and $\kappa = 3$ and compare the validation to a standard implementation of ESPRIT. As can be seen in the ESPRIT reconstruction in Figure 1 (top), it suffers from the presence of the outlier, as any parametric method would. The new method, illustrated in Figure 1 (bottom), deals with k decimated signals instead of the full signal and is more robust. In both approaches we choose $n = 20$. While the ESPRIT algorithm deals with a Hankel matrix of size 260×141 , the cluster analysis only needs Hankel matrices of size 70×30 . When the recording is corrupted by an outlier, here only one of the k decimated signals is affected. The decimated sample set that contains the outlier does not contribute to the formed clusters. But the cluster algorithm still detects clusters composed of at least $k - 1$ eigenvalues at the correct locations λ_i^k . Since one easily identifies the decimated signal that did not contribute to all clusters, the equations coming from that set of samples and contributing to (4) for the computation of the α_i , are best removed from the Vandermonde system.

6. ILLUSTRATION ON A HARMONIC SOUND

We consider a recorded sound of a guitar playing a D3 note corrupted by electrical humming [32], downloaded from the website freesound.org. The samples are collected at a rate of 48 kHz, for a duration of about 9 seconds in total (454 071 sample points t_j). We apply the method described above to audio windows of 1024 samples, with an overlap of 75% between the windows. The goal is to extract the sinusoidal tracks [33] that form the guitar partials. We choose the downsampling factor $k = 5$ and take $\kappa = 7$ (other combinations work as well, of course). So in each window the downsampled set contains 204 samples, which we use to extract $\nu = 61$ generalized eigenvalues, leaving us to Hankel matrices of size (at most) 143×61 . For the solution of (7) we use

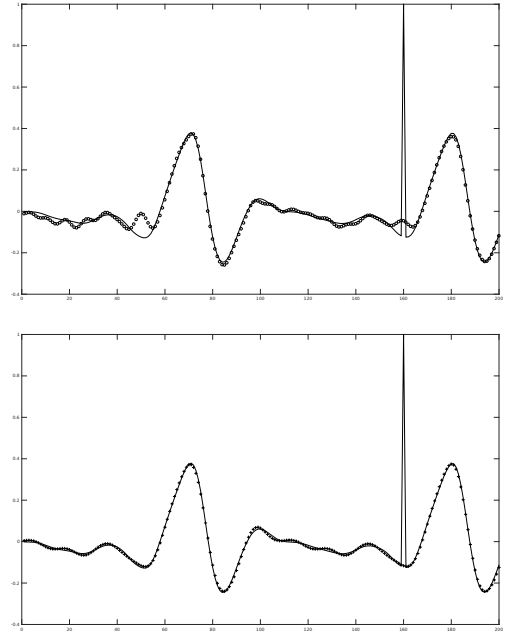


Figure 1: Corrupted trumpet recording reconstructed by ESPRIT (top, circles) and by the new method (bottom, crosses).

the ESPRIT algorithm [31]. After superimposing the $k = 5$ analyses of the downsampled audio windows, a cluster analysis using DBSCAN is performed for each window, thus retrieving the generalized eigenvalues λ_i^k most accurately.

To illustrate the regularization effect on the rectangular 143×61 analogon of (2) from choosing $k > 1$, we show in Figure 2 the distribution of the generalized eigenvalues $\lambda_i, i = 1, \dots, n$ of the full not downsampled 60-th windows starting at t_{15104} opposed to that of the $\lambda_i^5, i = 1, \dots, n$ of the downsampled set of samples from the same window.

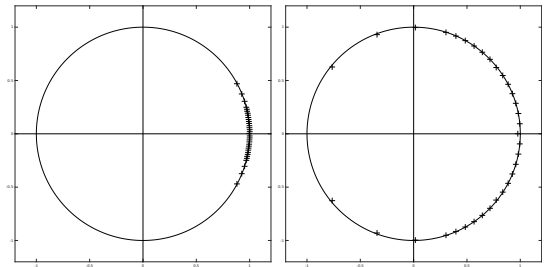


Figure 2: Generalized eigenvalues λ_i (left) versus λ_i^5 (right) from the 60-th window.

In the wake of the shift strategy discussed in Section 5 (we merely took $h = 1, \dots, 24 < H$), the value λ_i^κ can further be improved. After performing the cluster analysis on the superimposed results for λ_i^k , we can look at the λ_i^κ associated with each of these and compute their center of gravity (disregarding those that fall out of scope). Note that for the λ_i^κ no separate cluster analysis needs to be performed. The latter is illustrated in Figure 3 for $i = 16$, corresponding to the 16-th harmonic partial.

Since the technique is being applied to a harmonic sound, an

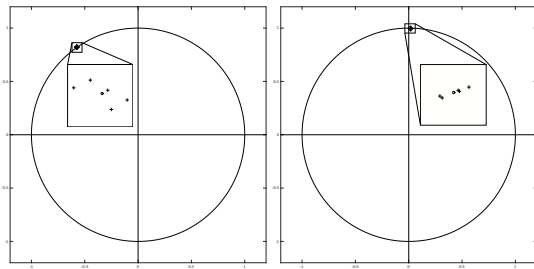


Figure 3: Cluster of λ_{15}^5 values (left) and λ_{15}^7 values (right).

additional step can be performed to estimate the base frequency (per window) more accurately. Once the stable frequencies $\phi_i, i = 1, \dots, n$ are retrieved, we look for a harmonic relation between them. We divide every detected harmonic partial ϕ_i by the integers $j = 1, \dots, 40$ (which is the largest number of partials expected) and we add these quotients to the discovered ϕ_i , in this way creating a new larger cluster at the base frequency, which we call ϕ_1 . The center of gravity of this larger cluster estimates the lowest partial of the harmonics. Using this estimate of the base frequency ϕ_1 , all higher harmonic partials $j\phi_1, j = -60, \dots, 60$ are reconstructed and substituted in one large rectangular 512×121 Vandermonde system (4), which serves as the coefficient matrix for the computation of the $\alpha_i, i = 1, \dots, 121$.

While moving from one window to the next over the course of the 9 seconds, the higher harmonic partials that are detected become weaker and fewer. So n decreases with time. We refer to Figure 4, where we again show the generalized eigenvalues λ_i and λ_i^5 , before and after regularization, now for one of the middle audio windows. Fortunately, the number of partials remains large enough during the whole audio fragment to rebuild the harmonics as described. Since the final reconstructed guitar sound only makes use of the ϕ_i from the stable generalized eigenvalues, the reconstruction, which can be downloaded from <https://www.uantwerpen.be/br-cu-ea-val-17/>, does not suffer from the electrical hum anymore.

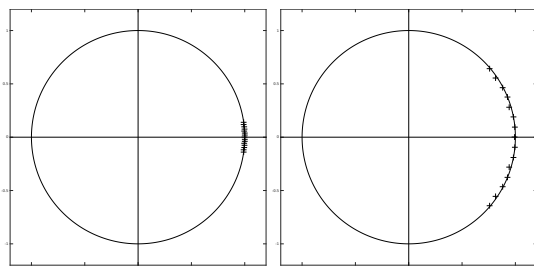


Figure 4: Generalized eigenvalues λ_i (left) versus λ_i^5 (right) from a middle window.

7. CONCLUSION

We present an approach that can be embedded in several parametric methods. We exploit the aliasing phenomenon caused by sub-Nyquist sampling and the connections with sparse interpolation and Padé approximation. The result is a parametric method that is able to discern between the stable and unstable components of

a multi-exponential model. It can thus remove outliers and/or the noisy part of the signal. We illustrate our approach on a harmonic sound where the validated output is used to refine the estimation of the lowest partial and reconstruct the signal thus eliminating the electrical humming present in the recording.

In our case study, we use a quasi-stationary sound and thus have not fully exploited the potential of the presented method yet. In the future we plan to apply the techniques described in Section 5 to decaying signals and signals with modulated amplitudes. Our illustrations pave the way for a wider field of unexplored applications and connections. Actually, every audio algorithm that makes use of a parametric method, may benefit from the ideas presented here.

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