

## MORE ON GRAPHS WITH JUST THREE DISTINCT EIGENVALUES

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*In honour of Dragoš Cvetković on the occasion of his 75th birthday.*

Let  $G$  be a connected non-regular non-bipartite graph whose adjacency matrix has spectrum  $\rho, \mu^{(k)}, \lambda^{(l)}$ , where  $k, l \in \mathbb{N}$  and  $\rho > \mu > \lambda$ . We show that if  $\mu$  is non-main then  $\delta(G) \geq 1 + \mu - \lambda\mu$ , with equality if and only if  $G$  is of one of three types, derived from a strongly regular graph, a symmetric design or a quasi-symmetric design (with appropriate parameters in each case).

### 1. INTRODUCTION

Let  $G$  be a graph of order  $n$  with  $(0, 1)$ -adjacency matrix  $A$ . An eigenvalue  $\sigma$  of  $A$  is said to be an eigenvalue of  $G$ , and  $\sigma$  is a *main* eigenvalue if the eigenspace  $\mathcal{E}_A(\sigma)$  is not orthogonal to the all-1 vector in  $\mathbb{R}^n$ . Always the largest eigenvalue, or *index*, of  $G$  is a main eigenvalue, and it is the only main eigenvalue if and only if  $G$  is regular. We say that  $G$  is an *integral* graph if every eigenvalue of  $G$  is an integer; and  $G$  is a *biregular* graph if it has just two different degrees. We use the notation of the monograph [6], where the basic properties of graph spectra can be found in Chapter 1.

Let  $\mathcal{C}_1$  be the class of connected graphs with just three distinct eigenvalues, and let  $\mathcal{C}_2$  be the class of connected graphs with exactly two main eigenvalues. It is an open problem to determine all the graphs in  $\mathcal{C}_1$ , and another open problem to determine all the graphs in  $\mathcal{C}_2$ . Here we continue the investigation of graphs in

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2010 Mathematics Subject Classification 05C50.

Keywords and Phrases: main eigenvalue, minimum degree, quasi-symmetric design, strongly regular graph, symmetric design.

$\mathcal{C}_1 \cap \mathcal{C}_2$  begun in [14]. Independently the authors of [3] investigated the biregular graphs in  $\mathcal{C}_1$ , and it is not difficult to see that these are precisely the graphs in  $\mathcal{C}_1 \cap \mathcal{C}_2$ : this follows from [3, Theorem 4.3(i)] and [14, Lemma 2.2]. Some examples of biregular graphs in  $\mathcal{C}_2 \setminus \mathcal{C}_1$  (of order 16) can be found in [10, Table 1].

In [14] it was noted that any graph  $G$  in  $\mathcal{C}_1 \cap \mathcal{C}_2$  is either integral or complete bipartite. Examples include the following, where  $G$  has spectrum  $\rho, \mu^{(k)}, \lambda^{(l)}$  with  $\rho > \mu > \lambda$ .

- (1)  $G$  is of *conical type*: here  $G$  is the cone over a strongly regular graph with parameters  $\lambda^2\mu + \lambda^2 - \lambda\mu, \mu - \lambda\mu, 2\mu + \lambda, \mu$  (see [12, 14]);
- (2)  $G$  is of *symmetric type*: here  $G$  is obtained from the incidence graph of a symmetric  $2$ -( $q^3 - q + 1, q^2, q$ ) design by adding all edges between blocks (see [7]);
- (3)  $G$  is of *affine type*: here  $G$  is obtained from the incidence graph of an affine  $2$ -( $q^3, q + 1, q^2, q$ ) design by adding all edges between intersecting blocks (see [8]).

There are infinitely many graphs of each type. The graphs of conical type include the cones  $K_1 \nabla H$ , where  $H$  is the Petersen graph ( $\mu = 1, \lambda = -2$ ), the Gewirtz graph ( $\mu = 2, \lambda = -4$ ) or a Chang graph ( $\mu = 4, \lambda = -2$ ). For graphs of symmetric type we have  $\rho = q^3, \mu = q - 1$  and  $\lambda = -q$ , and for graphs of affine type we have  $\rho = q^3 + q^2 + q, \mu = q$  and  $\lambda = -q$ . If  $G$  is a graph of any of these three types then  $\mu$  is the non-main eigenvalue, and the minimum degree  $\delta(G)$  is equal to  $1 + \mu - \lambda\mu$ .

To formulate a partial converse implicit in [14, Section 4] we extend the construction in (3) to any quasi-symmetric design which has 0 as one of its two intersection numbers. If  $\mathcal{D}$  is such a design then we denote by  $G_{\mathcal{D}}$  the graph obtained from the incidence graph of  $\mathcal{D}$  by adding all edges between intersecting blocks: thus  $G_{\mathcal{D}}$  is the union of the block graph of  $\mathcal{D}$  with the incidence graph of  $\mathcal{D}$ . We refer to the graphs  $G_{\mathcal{D}}$  as graphs of *quasi-symmetric type*. It follows from [14, Theorem 4.2] that if  $\mu$  is non-main and  $\delta(G) = 1 + \mu - \lambda\mu$  then  $G$  is of conical, symmetric or quasi-symmetric type; moreover if  $G$  is of quasi-symmetric type, with  $G = G_{\mathcal{D}}$ , then the parameters of  $\mathcal{D}$  are specified rational functions of  $\lambda$  and  $\mu$ , and the intersection numbers are 0 and  $-\lambda$ . (We have strong integrality conditions as a consequence.) In Section 3 we show conversely that for any quasi-symmetric design  $\mathcal{D}$  with these parameters and intersection numbers, the graph  $G_{\mathcal{D}}$  lies in  $\mathcal{C}_1 \cap \mathcal{C}_2$ , with  $\mu$  non-main and  $\delta(G_{\mathcal{D}}) = 1 + \mu - \lambda\mu$ . We shall see also that  $\delta(G) \geq 1 + \mu - \lambda\mu$  whenever  $G \in \mathcal{C}_1 \cap \mathcal{C}_2$  and  $\mu$  is non-main. (For an example with strict inequality here we may take  $G$  to be the unique maximal exceptional graph of order 36 [5, Section 6.1]: for this graph we have  $\delta(G) = 18, \mu = 5$  and  $\lambda = -2$ .)

To state one further observation, recall from [1] that a *multiplicative* graph is a connected graph whose distinct eigenvalues are  $\rho, \mu, -\mu$ , where  $\rho > \mu > 0$ . For such graphs,  $\lambda = -\mu$ , and in Section 4 we show that if  $\delta(G) = 1 + \mu + \mu^2$  then  $-\mu$  is a main eigenvalue. We deduce that the only multiplicative graphs in  $\mathcal{C}_1 \cap \mathcal{C}_2$  with  $\delta(G) = 1 + \mu + \mu^2$  are (a) the graphs of affine type, and (b) the cones over a strongly regular graph with parameters  $\mu^3 + 2\mu^2, \mu^2 + \mu, \mu, \mu$ . It was already shown in [1] that the graphs of type (b) lie in  $\mathcal{C}_1$ .

## 2. PRELIMINARIES

Here we note three results required in subsequent sections.

**Lemma 2.1** [14, Lemma 2.2]. *A graph  $G$  in  $\mathcal{C}_1 \cap \mathcal{C}_2$  has exactly two distinct degrees (say  $d_1, d_2$ ), and these degrees determine an equitable bipartition of  $G$ . Moreover, if  $G$  has spectrum  $\rho, \mu^{(k)}, \lambda^{(l)}$ , where  $\rho > \mu > \lambda$ , then  $d_h = \alpha_h^2 - \lambda\mu$ , where  $\alpha_h > 0$  ( $h = 1, 2$ ) and either*

- (a)  $\mu$  is non-main and  $\alpha_1\alpha_2 = -\lambda(\mu + 1)$ , or
- (b)  $\lambda$  is non-main and  $\alpha_1\alpha_2 = -\mu(\lambda + 1)$ .

Next recall that a quasi-symmetric design is a 2-design  $\mathcal{D}$  for which the number of points in the intersection of two blocks takes exactly two values. We say that  $\mathcal{D}$  is a quasi-symmetric  $(a, b, c, d, e)$ -design with intersection numbers  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ) if it has  $a$  points,  $b$  blocks, each point lies in  $c$  blocks, each block contains  $d$  points, any two points lie in exactly  $e$  blocks, and two blocks intersect in  $\alpha$  or  $\beta$  points. The *block graph* of  $\mathcal{D}$  has the blocks of  $\mathcal{D}$  as its vertices, with blocks adjacent if and only if they intersect in  $\beta$  points.

**Theorem 2.2** [11, Theorem 8.3.14]. *The block graph of a quasi-symmetric  $(a, b, c, d, e)$ -design with intersection numbers  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ) is strongly regular with spectrum  $\xi, \eta^{(a-1)}, \zeta^{(b-a)}$ , where*

$$\xi = \frac{(c-1)d - \alpha(b-1)}{\beta - \alpha}, \quad \eta = \frac{c - d - e + \alpha}{\beta - \alpha}, \quad \zeta = \frac{\alpha - d}{\beta - \alpha}.$$

To formulate [14, Theorem 4.2(c)] in terms of a quasi-symmetric design, we define

$$(1) \quad f_1(\lambda, \mu) = \frac{\lambda(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\mu + 1}, \quad f_2(\lambda, \mu) = \frac{(1 + \mu - \lambda\mu)(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\lambda(\mu + 1)}.$$

Now we have

**Theorem 2.3** [14, Theorem 4.2]. *Let  $G$  be a connected non-regular non-bipartite graph whose distinct eigenvalues are  $\rho, \mu, \lambda$  where  $\rho > \mu > \lambda$ . If  $\mu$  is non-main and  $\delta(G) = 1 + \mu - \lambda\mu$  then  $G$  is of conical, symmetric or quasi-symmetric type. Moreover, if  $G$  is of quasi-symmetric type then  $G = G_{\mathcal{D}}$  where  $\mathcal{D}$  is a quasi-symmetric  $(f_1(\lambda, \mu), f_2(\lambda, \mu), 1 + \mu - \lambda\mu, \lambda^2, 1 + \mu)$ -design with intersection numbers 0 and  $-\lambda$ .*

## 3. GRAPHS OF QUASI-SYMMETRIC TYPE

Our main objective in this section is to prove the converse of Theorem 2.3. It has already been noted that the hypotheses are satisfied if  $G$  is of conical or symmetric type. Accordingly suppose that  $G = G_{\mathcal{D}}$ , where  $\mathcal{D}$  is a quasi-symmetric

$(f_1(\lambda, \mu), f_2(\lambda, \mu), 1 + \mu - \lambda\mu, \lambda^2, 1 + \mu)$ -design with intersection numbers 0 and  $-\lambda$ . Note that  $\mu$  is a positive integer and  $\lambda$  is a negative integer. We show first that the distinct eigenvalues of  $G$  are  $\rho, \mu, \lambda$ , where  $\rho = -\lambda(1 + \mu - \lambda\mu)$ .

Let  $V_1$  be the set of blocks of  $\mathcal{D}$ , let  $V_2$  be the set of points of  $\mathcal{D}$ , and let  $G_1$  be the block graph of  $\mathcal{D}$ , that is, the graph induced by  $V_1$ . From Theorem 2.2 we know that  $G_1$  is strongly regular with eigenvalues  $\nu, \lambda + \mu, \lambda$ , where  $\nu = \lambda\mu(\lambda - 1)$ . Now let  $m_1(x) = (x - \nu)(x - \lambda - \mu)(x - \lambda)$ ,  $m(x) = (x - \rho)(x - \mu)(x - \lambda)$ . Let  $G$  have adjacency matrix  $A = \begin{pmatrix} A_1 & B^\top \\ B & O \end{pmatrix}$ , and let  $m(A) = \begin{pmatrix} X & Y^\top \\ Y & Z \end{pmatrix}$ , both partitioned in accordance with  $V_1 \cup V_2$ .

From the parameters and intersection numbers of  $\mathcal{D}$  we see that  $B^\top B = \lambda^2 I - \lambda A_1$  and  $BB^\top = -\lambda\mu I + (\mu + 1)J$ , where each entry of  $J$  is 1. On expanding  $m(A)$  and substituting for  $\rho$ ,  $B^\top B$  and  $BB^\top$ , we find that  $X = m_1(A_1)$ , and so  $X = O$ . Again,  $Y^\top = (A_1^2 - (\lambda + \mu + \nu)A_1 + (\lambda + \mu)\nu I)B^\top$ , from which it follows that  $Y^\top B = -\lambda m_1(A_1) = O$ . Since  $BB^\top$  is invertible, we have  $Y^\top = O$ .

Further calculation shows that  $Z = BA_1 B^\top - \mu(1 - \lambda + \lambda^2)(1 + \mu)J + \lambda\mu(\lambda + \mu)I$ . In what follows, we write  $\mathbf{j}$  for an all-1 vector, its length determined by context. We have  $BA_1 B^\top \mathbf{j} = \lambda^2 BA_1 \mathbf{j} = \lambda^2 \nu B \mathbf{j} = \lambda^3 \mu(\lambda - 1)(1 + \mu - \lambda\mu)\mathbf{j}$ , and it follows that

$$Z\mathbf{j} = \lambda^3 \mu(\lambda - 1)(1 + \mu - \lambda\mu)\mathbf{j} - \mu(1 - \lambda + \lambda^2)(1 + \mu)f_1(\lambda, \mu)\mathbf{j} + \lambda\mu(\lambda + \mu)\mathbf{j} = \mathbf{0}.$$

Now  $BB^\top$  commutes with  $A_1$  and hence with  $BA_1 B^\top$ . The matrices  $BB^\top$  and  $BA_1 B^\top$  are therefore simultaneously diagonalizable. Since  $\mathbf{j}$  is an eigenvector of both of these matrices, there exist  $f_1(\lambda, \mu)$  pairwise orthogonal eigenvectors, including  $\mathbf{j}$ , which are eigenvectors of both  $BB^\top$  and  $BA_1 B^\top$ . Let  $\mathbf{x}$  be such a vector orthogonal to  $\mathbf{j}$ . We have  $Z\mathbf{x} = BA_1 B^\top \mathbf{x} + \lambda\mu(\lambda + \mu)\mathbf{x}$  and  $BB^\top \mathbf{x} = -\lambda\mu\mathbf{x}$ . Now  $\lambda\mu B^\top \mathbf{x} = (B^\top B)B^\top \mathbf{x} = (\lambda^2 - \lambda)A_1 B^\top \mathbf{x}$ , whence  $A_1 B^\top \mathbf{x} = (\lambda + \mu)B^\top \mathbf{x}$  and  $BA_1 B^\top \mathbf{x} = -\lambda\mu(\lambda + \mu)\mathbf{x}$ . It follows that  $Z$  annihilates all  $f_1(\lambda, \mu)$  pairwise orthogonal eigenvectors, and hence that  $Z = O$ . We deduce that  $m(A) = 0$  and hence that the distinct eigenvalues of  $G$  are  $\rho, \lambda, \mu$  as required. Now we can prove:

**Theorem 3.1.** *Let  $G$  be a connected non-regular non-bipartite graph whose distinct eigenvalues are  $\rho, \mu, \lambda$  where  $\rho > \mu > \lambda$ . If  $\mu$  is non-main then  $\delta(G) \geq 1 + \mu - \lambda\mu$  with equality if and only if  $G$  is of conical, symmetric or quasi-symmetric type  $G_{\mathcal{D}}$ , where  $\mathcal{D}$  is a quasi-symmetric  $(f_1(\lambda, \mu), f_2(\lambda, \mu), 1 + \mu - \lambda\mu, \lambda^2, 1 + \mu)$ -design with intersection numbers 0 and  $-\lambda$ . (The functions  $f_1, f_2$  are defined in Eq.(1).)*

**Proof.** As noted in [7, 14], the adjacency matrix  $A$  of  $G$  satisfies the equation

$$(2) \quad (A - \mu I)(A - \lambda I) = \mathbf{a}\mathbf{a}^\top,$$

where  $A\mathbf{a} = \rho\mathbf{a}$  and each entry of  $\mathbf{a}$  is positive. By Lemma 2.1,  $G$  has two degrees, say  $d_1$  and  $d_2$ . Moreover  $d_h = \alpha_h^2 - \lambda\mu$  ( $h = 1, 2$ ) where, with a suitable ordering of vertices,  $\mathbf{a} = \begin{pmatrix} \alpha_1 \mathbf{j} \\ \alpha_2 \mathbf{j} \end{pmatrix}$ ,  $\alpha_1 > \alpha_2$  and  $\alpha_1 \alpha_2 = -\lambda(\mu + 1)$ . Let  $i, j$  be adjacent vertices with  $\deg(i) = d_1, \deg(j) = d_2$ . From Eq.(2), the number of  $i$ - $j$  walks of length 2 is

$\alpha_1\alpha_2 + \lambda + \mu$ , that is,  $-\lambda\mu + \mu$ . Now  $-\lambda\mu + \mu \leq \deg(j) - 1 = \delta(G) - 1$ , and so  $\delta(G) \geq 1 + \mu - \lambda\mu$ .

In the light of Theorem 2.3 and the argument above, it remains to show that when  $G = G_{\mathcal{D}}$ , the eigenvalue  $\mu$  is non-main and  $\delta(G) = 1 + \mu - \lambda\mu$ . In this case let  $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$  be the divisor matrix  $D$  determined by the equitable partition  $V_1 \dot{\cup} V_2$ , where  $V_1$  is the set of blocks of  $\mathcal{D}$  and  $V_2$  is the set of points of  $\mathcal{D}$ . We have  $r_{11} = \nu = \lambda\mu(\lambda - 1)$  and  $r_{22} = 0$ , and so  $D$  has trace  $\rho + \lambda$ . It follows from [14, Theorem 3.9.9] that  $G$  has exactly two main eigenvalues; moreover, since  $\rho$  is one main eigenvalue of  $G$ , the other is  $\lambda$ .

Finally, the vertices in  $V_1$  have degree  $\nu + \lambda^2 = -\lambda(-\lambda - \lambda\mu + \mu)$ , while those in  $V_2$  have the smaller degree  $1 + \mu - \lambda\mu$ . Thus  $\delta(G) = 1 + \mu - \lambda\mu$  and the proof of the theorem is complete.  $\square$

#### 4. FURTHER REMARKS

(1) A question which remains is whether there exist any graphs  $G_{\mathcal{D}}$  of quasi-symmetric type that are not of affine type. As noted in [14], any candidate has  $\lambda + \mu > 0$ ; moreover  $\lambda$  and  $\mu$  are not coprime. The candidate with smallest  $\mu - \lambda$  has order 625, with  $\mu = 9$  and  $\lambda = -6$ . By Theorem 3.1 such a graph exists if and only if there exists a quasi-symmetric  $(225, 400, 64, 36, 10)$ -design  $\mathcal{D}$ . Then  $G_1$ , the block graph of  $\mathcal{D}$ , is strongly regular with parameters 400, 378, 357, 360; according to [2], the existence of such a strongly regular graph is unknown. We note that the parameters of  $G_1$  and  $\mathcal{D}$  satisfy the four necessary conditions for the existence of  $\mathcal{D}$  formulated in [13, Theorem 6].

(2) The authors of [9] investigate the strongly regular graphs  $G$  with a *strongly regular decomposition*. This means that  $V(G) = V_1 \dot{\cup} V_2$ , where (for  $i = 1, 2$ ) the graph  $G_i$  induced by  $V_i$  is a strongly regular graph or a clique or a co-clique: then  $V_1 \dot{\cup} V_2$  is an equitable partition. In the case that one of  $G_1, G_2$  is a co-clique, the decomposition is said to be *improper*.

One can seek to generalize the results of [9] by requiring instead that  $G$  is a graph in  $\mathcal{C}_1$  with an equitable bipartition and an associated strongly regular decomposition. If  $G$  is not strongly regular then  $G \in \mathcal{C}_1 \cap \mathcal{C}_2$ , and the graphs of Theorem 3.1 are examples of such a graph with an improper strongly regular decomposition.

Now suppose that  $G$  is a graph in  $G \in \mathcal{C}_1 \cap \mathcal{C}_2$  having an improper strongly regular decomposition, with  $G_2$  a non-trivial co-clique and  $G_1$  a strongly regular graph with parameters  $n_1, r_1, e_1, f_1$ . We make two comments using the notation of Section 3. First,  $\mu$  is the non-main eigenvalue of  $G$  for otherwise  $-r_{12}r_{21} = \det(D) = \rho\mu > 0$ , a contradiction. Secondly, if  $i, j \in V_1$  ( $i \neq j$ ) then by Eq.(2) the  $(i, j)$ -entry of  $A^2$  is given by

$$a_{ij}^{(2)} = \begin{cases} \alpha_1^2 & \text{if } i \not\sim j, \\ \alpha_1^2 + \lambda + \mu & \text{if } i \sim j. \end{cases}$$

Similarly, if  $i, j \in V_2$  ( $i \neq j$ ) we have  $a_{ij}^{(2)} = \alpha_2^2$ . It follows that the neighbourhoods  $\{h \in V_2 : h \sim i\}$  ( $i \in V_1$ ) form either a symmetric 2-design or a quasi-symmetric  $(a, b, c, d, e)$ -design with intersection numbers  $\alpha, \beta$ , where  $a = n_2$ ,  $b = n_1$ ,  $c = r_{21}$ ,  $d = r_{12}$ ,  $e = \alpha_2^2$ ,  $\alpha = \alpha_1^2 - f_1$  and  $\beta = \alpha_1^2 + \lambda + \mu - e_1$  (cf. [9, Theorem 2.9]).

(3) Our final remark concerns multiplicative graphs, for which  $\lambda = -\mu$ . Let  $G$  be a connected non-regular graph with spectrum  $\rho, \mu^{(k)}, -\mu^{(l)}$ , where  $\rho > \mu > -\mu$  and one eigenvalue is non-main. Suppose that  $\delta(G) = 1 + \mu + \mu^2$ .

If  $-\mu$  is non-main, then we obtain a contradiction as follows. In the notation of Section 3, we have  $\alpha_2^2 = 1 + \mu$  as before, and  $\alpha_1\alpha_2 = \mu(\mu - 1)$  by Lemma 2.1. Then  $\alpha_1^2 = \mu^2(\mu - 1)^2/(\mu + 1)$ . Since  $\alpha_1 \neq 0$ , we deduce that  $\mu = 3$ . Then  $\alpha_1 = 3$ ,  $\alpha_2 = 2$ ,  $d_1 = 18$ ,  $d_2 = 13$ . Now  $d_2 < \rho < d_1$ , while  $\rho + k\mu + l(-\mu) = 0$ . Hence 3 divides  $\rho$ , and so  $\rho = 15$ .

From  $A\mathbf{a} = \rho\mathbf{a}$  we have  $\rho\alpha_1 = r_{11}\alpha_1 + r_{12}\alpha_2$  and  $\rho\alpha_2 = r_{21}\alpha_1 + r_{22}\alpha_2$ . Since also  $r_{11} + r_{12} = d_1$  and  $r_{21} + r_{22} = d_2$  we deduce that

$$r_{12} = \frac{\alpha_1(d_1 - \rho)}{\alpha_1 - \alpha_2} = 9, \quad r_{21} = \frac{\alpha_2(d_2 - \rho)}{\alpha_2 - \alpha_1} = 4.$$

Hence  $9n_1 = 4n_2$ , where  $n_h = |V_h|$  ( $h = 1, 2$ ). Also  $\|\mathbf{a}\|^2 = \rho^2 - \mu^2$ , whence  $9n_1 + 4n_2 = 216$ . We deduce that  $n_1 = 12$ ,  $n_2 = 27$ ,  $n = 39$ . From the equations  $15 + 3k - 3l = 0$ ,  $1 + k + l = 39$ , we obtain the contradiction  $6 + 2k = 39$ . This non-existence result can also be extracted from the results in [3].

We conclude that  $-\mu$  is a main eigenvalue, and so  $\mu$  is the non-main eigenvalue when  $G \in \mathcal{C}_1 \cap \mathcal{C}_2$ . We can then apply Theorem 3.1 with  $\lambda = -\mu$ . Then  $G$  is of conical or quasi-symmetric type. In the latter case,  $f_1(-\mu, \mu) = \mu^3$  and  $f_2(-\mu, \mu) = \mu^3 + \mu^2 + \mu$ ; moreover the block graph  $G_1$  is strongly regular with parameters  $\mu^3 + \mu^2 + \mu, \mu^3 + \mu^2, \mu^3 + \mu^2 - \mu, \mu^3 + \mu^2$ . Thus  $G_1 = (1 + \mu + \mu^2)K_\mu$ . It follows that  $G$  is of affine type because the blocks are partitioned into  $1 + \mu + \mu^2$  parallel classes, with blocks taken as parallel when they are disjoint (see [4, Theorem 1.44]). Accordingly we have:

**Proposition 4.1.** *let  $G$  be a connected graph whose distinct eigenvalues are  $\rho, \mu, -\mu$ , where  $\rho > \mu > -\mu$  and just one eigenvalue is non-main. Then  $\delta(G) = 1 + \mu + \mu^2$  if and only if  $G$  is either (a) a graph of affine type, or (b) the cone over a strongly regular graph with parameters  $\mu^3 + 2\mu^2, \mu^2 + \mu, \mu, \mu$ .*

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(Recived 13.09.2016.)

(Revised 11.11.2016.)