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STAR COMPLEMENTS AND EDGE-CONNECTIVITY IN FINITE GRAPHS

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Abstract

Let G be a finite graph with H as a star complement for a non-zero eigenvalue μ. Let $\kappa'(G)$, $\delta(G)$ denote respectively the edge-connectivity and minimum degree of G. We show that $\kappa'(G)$ is controlled by $\delta(G)$ and $\kappa'(H)$. We describe the possibilities for a minimum cutset of G when $\mu \notin \{-1, 0\}$. For such μ, we establish a relation between $\kappa'(G)$ and the spectrum of H when G has a non-trivial minimum cutset $E \nsubseteq E(H)$.

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1 Introduction

Let G be a finite simple graph with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix A of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. We use the notation of [8], where the basic properties of star sets and star complements are established in Chapter 5.

If G has H as a star complement of order t, for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$ [2, Theorem 2.3]. Thus there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1. In these circumstances, it is of interest to investigate properties of H that are reflected in G: connectedness is one such property, as observed in [11, Section 2]. It was shown in [13] that the vertex-connectivity $\kappa(G)$ is controlled by $\kappa(H)$ and the minimum degree $\delta(G)$. In particular, for each $k \in I\!N$, there exists a smallest non-negative integer $f(k)$ such that

$$\mu \notin \{-1, 0\},\ \kappa(H) \geq k,\ \delta(G) \geq f(k) \Rightarrow \kappa(G) \geq k.$$

Here we first establish an analogous result for edge-connectivity: for each $k \in I\!N$, there exists a smallest non-negative integer $g(k)$ such that

$$\mu \neq 0,\ \kappa'(H) \geq k,\ \delta(G) \geq g(k) \Rightarrow \kappa'(G) \geq k. \qquad (1)$$

The arguments for $\kappa'(G)$ are quite different from those for $\kappa(G)$, and rely on a property of dominating sets. Moreover, whereas little is known about the function f, we find that $g(1) = 0$ and $g(k) = k$ for all $k > 1$. (It was shown in [13] that $k \leq f(k) \leq \frac{1}{2}(k-1)(k+2)$, while $f(1) = 0$, $f(2) = 2,\ f(3) = 3,\ f(4) = 5,\ f(5) = 7$ and $f(6) \geq 8$.)

We go on to investigate the nature of minimum cutsets of G when $\mu \notin \{-1, 0\}$. Following [9], we say that such a cutset E is *trivial* if E consists of the edges containing a vertex v (necessarily of degree $\delta(G)$). The interesting case is that in which G has a nontrivial minimum cutset E not in $E(H)$, for then we can find an upper bound for $\kappa'(G)$ in terms of the spectrum of H. We note some consequences in the case that H is regular and μ is not a main eigenvalue.

2 Preliminaries

We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. The eigenvaues of G are denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$, in non-increasing order. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S, and $\Delta_S(u)$ for the S-neighbourhood $\{v \in S : v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. An all-1 vector is denoted by $\mathbf{j}$, its length determined by context.

The following result, known as the Reconstruction Theorem, is fundamental to the theory of star complements.

Theorem 2.1. (See [8, Theorem 5.1.7].) *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X.*
(i) *Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{2}$$

(ii) *If X is a star set for μ then $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in I\!\!R^k)$.*

Writing $H = G - X$, we see that the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$. Thus G is determined by μ, a star complement H for μ, and the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$. From Eq. (2) we have

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu \text{ if } u = v \\ -1 \text{ if } u \sim v \\ 0 \text{ otherwise.} \end{cases} \tag{3}$$

From Eq. (3) we deduce:

Lemma 2.2. (See [8, Proposition 5.1.4].) *Let X be a star set for μ in G, and let $H = G - X$.*
(i) If $\mu \neq 0$ then $V(H)$ is a dominating set in G.
(ii)If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set in G, that is, the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$ are non-empty and distinct.

We shall also need the following observation, which follows from the fact the multiplicity of an eigenvalue changes by 1 at most when a vertex is deleted (cf. [8, Corollary 1.3.12]).

Lemma 2.3. *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

The next lemma extends the result of [2] mentioned in Section 1.

Lemma 2.4. (See [13, Proposition 1.5(ii)] *Let G be a graph with X as a star set for μ, and let $H = G - X$. If $\mu \notin \{-1, 0\}$ and $|\cup_{i \in X} \Delta_H(i)| = d$, then $|X| \leq \binom{d+1}{2}$.*

Recall that μ is said to be a *main* eigenvalue of G if G has a μ-eigenvector not orthogonal to the all-1 vector in $I\!\!R^n$. From the description of $\mathcal{E}(\mu)$ in Theorem 2.1(ii), we have:

Lemma 2.5. (See [6, Proposition 0.3].) *The eigenvalue μ is non-main if and only if $\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{j} = -1$ for all $u \in X$.*

Recall that a (κ, τ)-*regular* set in a graph G is a set S of vertices such that (i) S induces a κ-regular subgraph, and (ii) every vertex not in S has τ neighbours in S. The following observation is implicit in [12, Proposition 1.5]; an alternative argument is given in [1, Theorem 3.2].

Lemma 2.6. *Let G be a graph with a κ-regular star complement H for the eigenvalue μ. Then μ is a non-main eigenvalue of G if and only if $V(H)$ is (κ, τ)-regular with $\tau = \kappa - \mu$.*

Proof. Let $H = G - X$. By Lemma 2.5, μ is a non-main eigenvalue if and only if $\mathbf{b}_u^T(\mu - \kappa)^{-1}\mathbf{j} = -1$ for all $u \in X$, equivalently $|\Delta_H(u)| = \kappa - \mu$ for all $u \in X$. $\square$

Several structural conditions sufficient to ensure that $\kappa'(G) = \delta(G)$ may be found in the survey paper [9]. An early example is the following result, due to Chartrand [3].

Lemma 2.7. *If the graph G has order $n \leq 2\delta(G) + 1$ then $\kappa'(G) = \delta(G)$.*

Finally we note a recent result of Cioabă [4] relevant to our consideration of regular star complements.

Theorem 2.8. (See [4, Theorem 1.3].) *Let k, s be integers such that $s \geq k \geq 2$, and let H be an s-regular graph of order t. If $\lambda_2(H) \leq s - \frac{(k-1)t}{(s+1)(t-s-1)}$ then $\kappa'(H) \geq k$.*

3 Edge-connectivity

Theorem 3.1. *Let $k \in \mathbb{N}$, and let G be a graph with H as a star complement for a non-zero eigenvalue μ. If $\kappa'(H) \geq k$ and $\delta(G) \geq k$ then $\kappa'(G) \geq k$.*

Proof. The result holds for $k = 1$ because $V(H)$ is a dominating set by Lemma 2.2(i). Accordingly we assume that $k > 1$ and suppose by way of contradiction that G has a cutset E with $|E| \leq k - 1$. Let $V(G) = U \,\dot\cup\, V$, where each edge in E joins U to V. If $V(H)$ meets both U and V then $\kappa'(H) < k$, contrary to assumption, and so without loss of generality, $V(H) \subseteq U$. Let $u_1, \ldots, u_p$ be the vertices of U adjacent to V, and let $V = \{v_1, \ldots, v_q\}$. Since $V(H)$ is a dominating set, each vertex of V is adjacent to U. Now we use an argument of Plesnik [10, Theorem 6].

Let $s_i = |\Delta_U(v_i)|$ $(i = 1, \ldots, q)$. Since v_i is adjacent to at most $q - 1$ vertices of V, we have

$$s_i + q - 1 \geq \deg(v_i) \geq k \geq |E| + 1 \quad (i = 1, \ldots, q).$$

Hence $\sum_{i=1}^q s_i + q(q-1) \geq q(|E| + 1)$, that is, $|E| + q(q-1) \geq q(|E| + 1)$. Hence $q(q-1) \geq (q-1)|E| + q$. Since $|E| \geq q$, this is a contradiction. $\square$

Corollary 3.2. *Let G be a graph with H as a star complement for a non-zero eigenvalue μ.*
(i) If $\kappa'(H) \geq \delta(G)$ then $\kappa'(G) = \delta(G)$.
(ii) If G is regular and $\kappa'(H) \geq k$ then $\kappa'(G) \geq k$.

Proof. (i) Applying Theorem 3.1 with $k = \delta(G)$, we have $\kappa'(G) \geq \delta(G)$. Always $\kappa'(G) \leq \delta(G)$, and so the result follows.
(ii) Here Theorem 3.1 applies because $\delta(G) \geq \delta(H) \geq \kappa'(H) \geq k$. $\square$

We see that $\kappa'(G)$ is controlled by $\kappa'(H)$ and $\delta(G)$; explicitly, we have $\min\{\kappa'(H), \delta(G)\} \leq \kappa'(G) \leq \delta(G)$. Moreover, for each $k \in \mathbb{N}$, there exists a least non-negative integer $g(k) \leq k$ such that the relation (1) holds. The following example shows that $g(k) = k$ for all $k > 1$. (We know already that $g(1) = 0$, because G is connected whenever H is connected.)

Example 3.3. For $k \geq 2$, let G_k be the graph obtained from a $(k+1)$-clique H_k by adding a vertex of degree $k-1$, and let $\mu = \lambda_1(G_k)$. Since G_k is connected, we have $\mu > \lambda_1(H_k)$, and so H_k is a star complement for μ in G_k. Now $\kappa'(G_k) = k-1 = \delta(G_k)$, while $\kappa'(H_k) = k$. Hence $g(k) \geq k$. □

In what follows, we investigate the situations in which a strict inequality holds in the hypotheses of Theorem 3.1, that is either (a) $\kappa'(H) \geq k$ and $\delta(G) \geq k+1$ (see Corollary 3.5), or (b) $\kappa'(H) \geq k+1$ and $\delta(G) \geq k$ (see Corollary 3.7).

Proposition 3.4. *Let G be a graph with H as a star complement for a non-zero eigenvalue μ. If $\kappa'(H) = \kappa'(G)$ and G has a minimum cutset $E \not\subseteq E(H)$ then $\kappa'(G) = \delta(G)$.*

Proof. We define U and V as in Theorem 3.1. Again we may take $V(H) \subseteq U$, for otherwise H can be disconnected by removing the edges in $E \cap E(H)$. If $\kappa'(G) < \delta(G)$ then we have

$$s_i + q - 1 \geq \deg(v_i) \geq \delta(G) \geq |E| + 1 \quad (i = 1, \ldots, q),$$

and we obtain a contradiction as before. □

Corollary 3.5 *Let G be a graph with H as a star complement for a non-zero eigenvalue μ. If $\kappa'(H) \geq k$ and $\delta(G) \geq k+1$ then either (a) $\kappa'(G) \geq k+1$ or (b) $\kappa'(H) = \kappa'(G) = k$ and every minimum cutset of G lies in $E(H)$.*

Proof. By Theorem 3.1, we have $\kappa'(G) \geq k$; moreover, (a) holds if $\kappa'(H) \geq k+1$. If $\kappa'(G) = k$ then $\kappa'(H) = k$, and (b) holds by Proposition 3.4. □

We remark in passing that if E is a non-trivial cutset of G in $E(H)$ then the multiplicity of μ is subject to an upper bound which improves that given in [2, Theorem 2.3]. This last result says that if $H = G - X$ of order $t > 4$ then $|X| \leq \binom{t}{2}$. On the other hand, if $V(H) = V_1 \dot\cup V_2$, where each edge in E joins V_1 to V_2, let $|V_i| = t_i$, $X_i = \{u \in X : \Delta_H(u) \subseteq V_i\}$ $(i = 1, 2)$. Then $t = t_1 + t_2$, where $t_1 \geq 2$ and $t_2 \geq 2$ because E is non-trivial. Moreover, $X = X_1 \dot\cup X_2$ and by Lemma 2.3 we may apply Lemma 2.4 to $G - X_1$, $G - X_2$ to deduce that $|X| \leq \binom{t_1+1}{2} + \binom{t_2+1}{2}$. Now when $t_1 \geq 2$ and $t_2 \geq 2$ we have $\binom{t_1+1}{2} + \binom{t_2+1}{2} \leq \binom{t_1+t_2}{2}$, with strict inequality unless $t_1 = t_2 = 2$.

We say that a set E of edges in G is a *k-clique matching* if E consists of independent edges u_iv_i $(i = 1, \ldots, k)$ such that the vertices $v_1, \ldots, v_k$ induce a clique which is a component of $G - E$. (Note that, in a connected graph, a 1-clique matching is a trivial minimum cutset consisting of a pendant edge.) For a vertex v of G we write $E(v)$ for the set of edges containing v.

Proposition 3.6. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. If $\kappa'(H) \geq \kappa'(G) = k$ and E is a minimum cutset of G then one of the following holds:*
(a) $\kappa'(H) = k$ and $E \subseteq E(H)$;
(b) $E = E(v)$ for some $v \notin V(H)$;
(c) $E \cap E(H) = \emptyset$ and E is a k-clique matching.

Proof. Note first that if $E \subseteq E(H)$ then $\kappa'(H) = k$, and (a) holds. Now suppose that $E \not\subseteq E(H)$. Since $\kappa'(H) \geq k$ we have $E \cap E(H) = \emptyset$, and we may define U, V as before, with $V(H) \subseteq U$. In the notation of Theorem 3.1 we have

$$s_i + q - 1 \geq \deg(v_i) \geq \delta(G) \geq \kappa'(G) = |E| \quad (i = 1, \ldots, q), \tag{4}$$

whence $\sum_{i=1}^{q} s_i + q(q-1) \geq q|E|$, that is,

$$q(q-1) \geq (q-1)|E|.$$

If $q = 1$, we have case (b). If $q > 1$ then $q \geq |E|$ and necessarily $q = |E| = k$. In this situation, $\sum_{i=1}^{q} s_i = q$ and so all s_i are equal to 1. By Lemma 2.2(ii), the edges in E are independent. Since equality holds throughout Eq. (4), the vertices $v_1, \ldots, v_q$ induce a clique, and so E is a k-clique matching. □

Corollary 3.7. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$. If $\kappa'(H) \geq k+1$ and $\delta(G) \geq k$ then either (a) $\kappa'(G) \geq k+1$ or (b) $\kappa'(G) = k$ and every nontrivial minimum cutset of G is a k-clique matching.*

Proof. We first apply Theorem 3.1: if $\delta(G) \geq k+1$ then $\kappa'(G) \geq k+1$, and if $\delta(G) = k$ then $\kappa'(G) = k$. In the latter case, (b) follows from Proposition 3.6 because a cutset of size k cannot lie in $E(H)$. □

Next we investigate case (c) of Proposition 3.6: we establish a connection between k and the spectrum of H when $\kappa'(H) \geq k$ and G has a k-clique matching $E \not\subseteq E(H)$. Recall that the condition $\kappa'(H) \geq k$ ensures that $E \cap E(H) = \emptyset$.

Theorem 3.8. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$, with $\kappa'(H) \geq \kappa'(G) = k > 1$. Suppose that G has a k-clique matching $E \not\subseteq E(H)$, and let $\nu_1, \nu_2, \ldots, \nu_t$ be the eigenvalues of H in non-increasing order.*

(i) If $\mu > -1$ then there exists a smallest h such that $\nu_h < \mu$. In this case, $t \geq h + k - 2$ and $\nu_{h+k-2} \geq \mu - \frac{1}{\mu+1}$.

(ii) If $\mu < -1$ then there exists a largest m such that $\nu_m > \mu$. In this case, $m \geq k$, $\nu_{m-k+1} \leq \mu - \frac{1}{\mu+1}$ and $\nu_m \leq \mu + \frac{1}{k-\mu-1}$.

Proof. As before, we let $V(G) = U \,\dot\cup\, V$, where each edge in E joins U to V and $V(H) \subseteq U$. We apply Theorem 2.1 to the graph obtained from G by deleting the vertices in $U \setminus V(H)$; by Lemma 2.3, H remains a star complement for μ. We have

$$(\mu+1)I - J = B^\top(\mu I - C)^{-1}B \tag{5}$$

where J is the all-1 matrix of size $k \times k$, C is the adjacency matrix of H and without loss of generality, $B^\top = (I|O)$. Since $B^\top B = I$, the eigenvalues of $(\mu+1)I - J$ interlace those of $(\mu I - C)^{-1}$ in accordance with [8, Theorem 1.3.11].

(i) *The case* $\mu > -1$. If $\nu_t > \mu$ then all eigenvalues of H exceed -1 and so $H = \overline{K_t}$, a contradiction since $\kappa'(H) \geq 2$. Hence there is a smallest h

such that $\nu_h < \mu$, and $(\mu I - C)^{-1}$ has $t-h+1$ positive eigenvalues, namely $(\mu-\nu_h)^{-1}, (\mu-\nu_{h+1})^{-1}, \ldots, (\mu-\nu_t)^{-1}$ in non-increasing order. Now the $k-1$ largest eigenvalues of $(\mu+1)I-J$ are all equal to $\mu+1$. By interlacing, $\lambda_i((\mu I-C)^{-1}) \geq \mu+1 > 0$ $(i=1,\ldots,k-1)$, and so $t-h+1 \geq k-1$; moreover $\mu+1 \leq (\mu-\nu_{h+k-2})^{-1}$. equivalently, $\nu_{h+k-2} \geq \mu - \frac{1}{\mu+1}$.

(ii) *The case* $\mu < -1$. Since $\mu < 0$, we have $\nu_1 > \mu$ and so there is a largest m such that $\nu_m > \mu$. The negative eigenvalues of $(\mu I - C)^{-1}$ are $(\mu-\nu_m)^{-1}, (\mu-\nu_{m-1})^{-1}, \ldots, (\mu-\nu_1)^{-1}$ in non-decreasing order, while $(\mu+1)I-J$ has k negative eigenvalues. Hence $m \geq k$. By interlacing, $\lambda_{t-k+1}((\mu I-C)^{-1}) \leq \lambda_1((\mu+1)I-J)$, that is, $(\mu-\nu_{m-k+1})^{-1} \leq \mu+1$, equivalently $\nu_{m-k+1} \leq \mu - \frac{1}{\mu+1}$. Also by interlacing, $\lambda_t((\mu I-C)^{-1}) \leq \lambda_k((\mu+1)I-J)$, that is, $(\mu-\nu_m)^{-1} \leq \mu+1-k$, equivalently $\nu_m \leq \mu + \frac{1}{k-\mu-1}$. □

Corollary 3.9. *Let G be a graph with H as a star complement for an eigenvalue $\mu \notin \{-1, 0\}$, with $\kappa'(H) \geq \kappa'(G) = k > 1$. Suppose that G has a k-clique matching $E \not\subseteq E(H)$.*

(i) If $\mu > -1$ then $k-1$ is bounded above by the number of eigenvalues of H in the interval $[\mu - \frac{1}{\mu+1}, \mu)$.

(ii) If $\mu < -1$ then k is bounded above by the number of eigenvalues of H in the interval $(\mu, \mu - \frac{1}{\mu+1}]$.

We see that (in the situation of Corollary 3.9) for any $\mu \notin \{-1, 0\}$, we have $\kappa'(G) \leq 1 + e_H(\mu)$, where $e_H(\mu)$ is the number of eigenvalues of H between μ and $\mu - \frac{1}{\mu+1}$ inclusive. We shall give an example in which this bound is attained for $\kappa'(G) = 3$. The example arises in the context of the following result, where we apply Theorem 3.8 in the case that H is regular and μ is non-main.

Theorem 3.10. *Let G be a graph with the s-regular graph H as a star complement for the non-main eigenvalue μ. Let $\kappa'(G) = k$, where $s \geq k > 1$ and let $\nu_1, \nu_2, \ldots, \nu_t$ be the eigenvalues of H in non-increasing order. Suppose that G has a k-clique matching $E \not\subseteq E(H)$. Then $\mu = s-1$ and the following hold.*

(i) If $\nu_2 \leq s - \frac{2(k-1)}{s+1}$ then $\kappa'(H) \geq k$.

(ii) If $\kappa'(H) \geq k$, $l > 2$ and $\nu_l < s - 1 - \frac{1}{s}$ then $k \leq l-1$.

Proof. We use the notation of Theorem 3.8. Let G^* be the graph obtained from G by deleting the vertices in $U \setminus V(H)$. Note that μ remains a non-main eigenvalue of G^* by Lemma 2.5. Since $V(H)$ is an $(s,1)$-regular set in G^*, we have $\mu = s-1$ by Lemma 2.6.

Assertion (i), due to Cioabă [4], holds whether or not μ is a main eigenvalue. Indeed, if H has order $t \leq 2s+1$ then $\kappa'(H) = s \geq k$ by Lemma 2.7, while if $t \geq 2s+2$ then $\nu_2 \leq s - \frac{2(k-1)}{s+1} \leq s - \frac{(k-1)t}{(s+1)(t-s-1)}$ and we have $\kappa'(H) \geq k$ by Theorem 2.8.

For (ii) we note that $\nu_1 = s$, and so if $\nu_l < s-1-\frac{1}{s}$ then H has at most $l-2$ eigenvalues in the interval $[s-1-\frac{1}{s}, s-1)$. By Corollary 3.9(i), we have $k-1 \leq l-2$. □

We illustrate Theorem 3.10 with the following example, found experimentally using the computer package GRAPH [5].

Example 3.11. Let H be the 3-regular graph of order 10 which appears as the second graph in the list of exceptional regular graphs given in [7, Appendix A3.3]. Let G be the graph obtained from H by adding a 3-clique and the 3-clique matching E shown in Fig. 1, where the edges in E join white vertices to black. Note that $\kappa'(H) = 3$, either directly or by Theorem 3.10(i), and $\kappa'(G) = 3$ by construction. The spectrum of G is

$$3.2731, 2^{(3)}, 0.8596, 0^{(2)}, -1^{(2)}, -2^{(3)}, -2.1326,$$

where non-integer eigenvalues are given to four decimal places. The spectrum of H is

$$3, 1.8794^{(2)}, 1, -0.3473^{(2)}, -1.5321^{(2)}, -2^{(2)},$$

and so H is a star complement for 2. By Lemma 2.6, 2 is a non-main eigenvalue of G. In the notation of Theorem 3.10, we have $1 = \nu_4 < s-1-\frac{1}{s} = \frac{5}{3}$, and so the bound in Theorem 3.10(ii) is sharp for $l = 4$. Similarly, the bound in Corollary 3.9(i) is sharp for $\mu = 2$. □

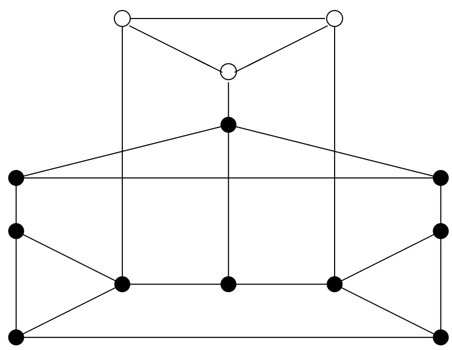

Fig. 1. The graph of Example 3.12.

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