ON INDUCED MATCHINGS AS STAR COMPLEMENTS IN REGULAR GRAPHS

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Abstract

We determine all the finite regular graphs which have an induced matching or a cocktail party graph as a star complement.

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1 Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k. (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix of G has dimension k.) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. The fundamental properties of star sets and star complements are established in [5, Chapter 5]. A survey of star complements in regular graphs may be found in [9], along with a description of the regular graphs with a star or windmill as a star complement. Here we first determine all the regular graphs with an induced matching as a star complement. It turns out that in each case, the star set X and its complement $\bar{X}$ form an equitable bipartition of the vertex set $V(G)$; equivalently, X and $\bar{X}$ are regular sets in the sense of [3, 6]. The motivation for our investigation is the example of the Petersen graph, which has $3K_2$ as a star complement for the eigenvalue -2. This example was noted in the context of regular sets by Cardoso [10, Problem AWG12], and in the context of star complements by the author [7, Example 6]. We shall see in Section 2 that the only other connected examples are a 3-cycle and the complete bipartite graphs $K_{r,r}$. In Section 3, we use our results to find the regular graphs with a cocktail party graph as a star complement.

We use the terminology of [5]. We write $\overline{G}$ for the complement of G, G_X for the subgraph of G induced by X, and '$u \sim v$' to mean that vertices u and v are adjacent. We shall require the following result.

Theorem 1.1 [5, Theorem 5.1.7] *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix* $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, *where A_X is the adjacency matrix of G_X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{1}$$

In this situation, $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in I\!R^k)$.

If $H = G - X$, the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$.

We define a bilinear form on $I\!R^{n-k}$ by:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top(\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in I\!R^{n-k}).$$

By equating entries in (1) we see that X is a star set for μ if and only if μ is not an eigenvalue of $G - X$ and the following conditions hold:

$$\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu, \quad \text{for all} \quad u \in X, \tag{2}$$

and for distinct $u, v \in X$,

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1 \text{ if } u \sim v, \quad \langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0 \text{ if } u \not\sim v. \tag{3}$$

In view of Equations (2) and (3), we have:

Proposition 1.2 [5, Proposition 5.1.4] *Let X be a star set for μ in G and let $H = G - X$.*
(i) *If $\mu \neq 0$ then the H-neighbourhoods of vertices in X are non-empty.*
(ii) *If $\mu \neq -1, 0$ then the H-neighbourhoods of vertices in X are distinct and non-empty.*

If G is r-regular and $\mu \neq r$ then the all-1 vector $\mathbf{j}_n$ is orthogonal to $\mathcal{E}(\mu)$; in other words, μ is a non-main eigenvalue (see [8], for example). From the description of $\mathcal{E}(\mu)$ in Theorem 1.1, we have the following result, where we write $\mathbf{j}$ for $\mathbf{j}_{n-k}$.

Proposition 1.3 [4, Proposition 0.3] *With the notation above, μ is a non-main eigenvalue if and only if*

$$\langle \mathbf{b}_u, \mathbf{j} \rangle = -1 \quad \text{for all } u \in X. \tag{4}$$

2 Induced matchings

We suppose first that G is a connected r-regular graph with a star complement H for μ of the form hK_2 ($h \in \mathbb{N}$). Note that $\mu \neq \pm 1$. If $\mu = r$ then $k = 1$ since G is connected (see [5, Corollary1.3.8]). Since a vertex of H is adjacent to the unique vertex in X, we have $r = 2$ and then G is a 3-cycle. Accordingly, we suppose that $\mu \neq r$, and invoke Proposition 1.3.

We retain the notation of Section 1 and consider $\Delta_H(u)$ for arbitrary fixed $u \in X$. We may take C to be block-diagonal with each block equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The subgraph of H induced by $\Delta_H(u)$ has the form $aK_2 \dot\cup bK_1$. Since $(\mu I - C)^{-1} = (\mu^2 - 1)^{-1}(\mu I + C)$, equations (2) and (4) yield:

$$\mu = \frac{1}{\mu^2 - 1}\{\mu(2a + b) + 2a\}, \tag{5}$$

$$-1 = \frac{1}{\mu^2 - 1}\{\mu(2a + b) + (2a + b)\}. \tag{6}$$

Solving (5) and (6), we find that

$$2a = \mu(\mu - 1)(\mu + 2), \quad b = -(\mu - 1)(\mu + 1)^2. \tag{7}$$

Since $\mu = 1 - 2a - b$, we see that μ is a non-positive integer.

We deal first with the case $\mu = 0$. Here $a = 0$, $b = 1$ and we write u' for the unique neighbour of u in H. If v is a neighbour of u in X then it follows from (3) that the vertices u, v, u', v' induce a 4-cycle. Hence each vertex in the component of G_X containing u is adjacent to u' or v'. (In fact, if w is a vertex of X such that $u \neq w \sim v$, then $w' = u'$ and $w \not\sim u$.) Since G is connected, necessarily $h = 1$. Then $k = n-2$ and the spectrum of G has the form $-\lambda$, $0^{(n-2)}$, λ. Hence G is the complete bipartite graph $K_{r,r}$ (cf. [5, Theorem 3.2.4]).

Now we may assume that μ is a negative integer. Since $a \geq 0$, it follows from (7) that $\mu = -2$, and hence that $a = 0$, $b = 3$, $h \geq 3$. By interlacing [5, Corollary 1.3.12], -2 is the least eigenvalue of G. Now G is not a generalized line graph because the induced subgraph $H + u$ has a component isomorphic to the subdivided star $S(K_{1,3})$ (see [4, Theorem 2.3.18]). Thus G is an exceptional graph, as defined in [4]. The 187 exceptional regular graphs were determined in [2], and they are listed in [4, Table A3.3]. They are partitioned into three 'layers': the graphs in the first, second, third layer have $n = 2(r+2)$, $n = \frac{3}{2}(r+2)$, $n = \frac{4}{3}(r+2)$ respectively (see [1, Theorem 3.12.2] or [4, Theorem 4.1.5]). Any exceptional graph may be represented in the root system E_8, in the sense of [4, Chapter 3]. In such a representation, h independent edges determine h pairwise orthogonal two-dimensional subspaces of $I\!R^8$, and so $h \leq 4$.

Now we count edges between X and $\bar{X}$. If $h = 4$ then we have

$$8(r-1) = 3(n-8) \leq 6(r+2) - 24 = 6r - 12,$$

a contradiction. If $h = 3$ and G lies in the second or third layer, then

$$6(r-1) = 3(n-6) \leq \frac{9}{2}(r+2) - 18 = \frac{9}{2} - 9,$$

another contradiction. Hence G lies in the first layer and has -2 as an eigenvalue of multiplicity $n - 6$. The only such graph is the Petersen graph (numbered 5 in [4, Table A3.3]). Gathering together our conclusions in the cases $\mu = r$, $\mu = 0$, $\mu \notin \{r, 0\}$, we have the following result.

Theorem 2.1 *If G is a connected r-regular graph ($r > 0$) with hK_2 ($h > 0$) as a star complement for the eigenvalue μ, then one of the following holds:*
(a) *$r = 2$, $h = 1$, $\mu = 2$ and G is a 3-cycle;*
(b) *$h = 1$, $\mu = 0$ and $G = K_{r,r}$;*
(c) *$r = 3$, $h = 3$, $\mu = -2$ and G is the Petersen graph.*

Conversely, each of the graphs in (a),(b),(c) satisfies the hypotheses of Theorem 2.1. Finally, we drop the requirement that G is connected. Suppose that G is r-regular with components $G_1, \ldots, G_m$, where G_i contains h_i of the components of H ($i = 1, \ldots, m$). Since $r > 0$, G_i is r-regular with $h_i K_2$ ($h_i > 0$) as a star complement for μ. The cases (a), (b), (c) of Theorem 2.1 are distinguished by the value of μ, and so we have the following.

Corollary 2.2 *Let μ be an eigenvalue of the r-regular graph G ($r > 0$). Then G has hK_2 ($h > 0$) as a star complement for μ if and only if one of the following holds:*
(a) *$r = 2$, $\mu = 2$ and $G = hK_3$;*
(b) *$\mu = 0$ and $G = hK_{r,r}$;*
(c) *$r = 3$, $\mu = -2$, $h = 3m$ ($m \in I\!N$) and $G = mP$, where P denotes the Petersen graph.*

3 Cocktail party graphs

Here we suppose that G is an r-regular graph with a star complement H for μ of the form $\overline{hK_2}$ ($h \in I\!N$). Note that H has spectrum $-2^{(h-1)}, 0^{(h)}, 2h-2$. Moreover, if $h > 1$ then H is connected and so G is connected by Proposition 1.2(i). It is feasible to use the method of Section 2 to determine the possible graphs G, but the calculations are cumbersome and it is more efficient to use the following observation.

Proposition 3.1 *Let G be an r-regular graph with an s-regular graph H of order t as a star complement for the eigenvalue μ. If $\mu \notin \{s-t, r\}$ then $\overline{H}$ is a star complement for $-1-\mu$ in $\overline{G}$.*

Proof. Here, μ is a non-main eigenvalue of G, and so if μ has multiplicity k in G then $-1-\mu$ has multiplicity at least k in $\overline{G}$.

Suppose by way of contradiction that $-1-\mu$ is an eigenvalue of $\overline{H}$. We have $-1-\mu \neq t-s-1$, and so $-1-\mu$ is a non-main eigenvalue of $\overline{H}$. Then μ is an eigenvalue of H, a contradiction.

Since $-1-\mu$ is not an eigenvalue of $\overline{H}$, the multiplicity of $-1-\mu$ as an eigenvalue of $\overline{G}$ is exactly k, by interlacing. Hence $\overline{H}$ is a star complement for $-1-\mu$ in $\overline{G}$. □

Theorem 3.2 *Let μ be an eigenvalue of the r-regular graph G ($r > 0$). Then G has $\overline{hK_2}$ ($h > 0$) as a star complement for μ if and only if one of the following holds:*
(a) *$\mu = 1$, $h = 1$, $G = 2K_2$ and $r = 1$;*
(b) *$\mu = -1$, $G = \overline{hK_{q,q}}$ and $r = 2qh - q - 1$;*
(c) *$\mu = 1$, $h = 3m$ ($m \in I\!N$), $G = \overline{mP}$ (where P is the Petersen graph) and $r = 10m - 4$.*

Proof. Let H be a star complement for μ in G, with $\overline{H} = hK_2$. Suppose first that $h > 1$; then $\mu \neq -2$. If also $\mu \neq r$ then by Proposition 3.1, $\overline{H}$ is a star complement for $-1-\mu$ in $\overline{G}$, and we apply Corollary 2.2 to $\overline{G}$. Thus G has one of the forms $\overline{hK_3}$, $\overline{hK_{q,q}}$, $\overline{mP}$, with $\mu = -3, -1, 1$ respectively. Only the second and third possibilities arise here, and they feature in cases (b) and (c) of the Theorem. If $\mu = r$ then μ is a simple eigenvalue of G because G is connected, and so G has order $2h+1$, with $r > 2h-2$. In this situation, $G = K_{r+1}$, a contradiction.

It remains to consider the case $h = 1$. Let $H = G - X$, and fix $u \in X$. Since $\mu \neq 0$, we know from Proposition 1.2(i) that either (1) $\mu = \pm\sqrt{2}$ and $H + u = K_{1,2}$, or (2) $\mu = \pm 1$ and $H + u = K_2 \dot\cup K_1$. Now either (1) holds for all $u \in X$, or (2) holds for all $u \in X$. In the first case, we obtain the contradiction $G = K_{1,2}$ from Proposition 1.2(ii). In the second case, non-adjacent vertices u, v of X cannot have a common neighbour in H (for otherwise $H + u + v$ does not have ± 1 as an eigenvalue). Thus each component containing a vertex of H is complete, and we have $G = 2K_{r+1} = \overline{K_{r+1,r+1}}$. Two possibilities arise: either $\mu = 1$ and $r = 1$, or $\mu = -1$ and r is arbitrary. These possibilities appear in cases (a) and (b) of the Theorem. □

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