

Rowlinson P (2014) Star complements and connectivity in finite graphs, *Linear Algebra and Its Applications*, 442, pp. 92-98.

This is the peer reviewed version of this article

NOTICE: this is the author's version of a work that was accepted for publication in Linear Algebra and its Applications resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Linear Algebra and its Applications, [VOL 442 (2014)] DOI: <http://dx.doi.org/10.1016/j.laa.2013.06.021>

STAR COMPLEMENTS AND CONNECTIVITY IN FINITE GRAPHS

Peter Rowlinson¹

Mathematics and Statistics Group
Institute of Computing Science and Mathematics
University of Stirling
Scotland FK9 4LA

Abstract

Let G be a finite graph with H as a star complement for an eigenvalue other than 0 or -1 . Let $\kappa(G)$, $\delta(G)$ denote respectively the vertex-connectivity and minimum degree of G . We prove that $\kappa(G)$ is controlled by $\delta(G)$ and $\kappa(H)$. In particular, for each $k \in \mathbb{N}$ there exists a smallest non-negative integer $f(k)$ such that $\kappa(G) \geq k$ whenever $\kappa(H) \geq k$ and $\delta(G) \geq f(k)$. We show that $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$ and $f(5) = 7$.

AMS Classification: 05C50

Keywords: graph, connectivity, eigenvalue, star complement.

¹Tel.: +44 1786 467468; fax +44 1786 464551; email: p.rowlinson@stirling.ac.uk

1 Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k . (Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$ -adjacency matrix A of G has dimension k .) A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G . We use the notation of [7], where the fundamental properties of star sets and star complements are established in Chapter 5.

It is well known that if $\mu \neq -1$ or 0 and $n > 4$ then $|X| \leq \binom{n-k}{2}$ [1]; in particular, there are only finitely many graphs with a prescribed star complement H for some eigenvalue other than 0 or -1 . Certain graphs can be characterized by a star complement: for surveys, see [9] and [11]. More generally, it is of interest to investigate properties of H that are reflected in G : connectedness is one such property, as noted in [8, Section 2]. Here we discuss k -connectedness for $k > 1$. In Section 2 we show that for each $k \in \mathbb{N}$ there exists a non-negative integer $F(k)$ with the following property: if $\mu \notin \{-1, 0\}$, H is k -connected and G has least degree $\delta(G) \geq F(k)$ then G is k -connected. It is straightforward to show that if $f(k)$ is the smallest non-negative integer with this property, then $f(1) = 0$ and $f(2) = 2$. In Sections 3 and 4 we show that $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $8 \leq f(6) \leq 20$.

We take $V(G) = \{1, \dots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. For $S \subseteq V(G)$, we write G_S for the subgraph induced by S , and $\Delta_S(u)$ for the S -neighbourhood $\{v \in S : v \sim u\}$. For the subgraph H of G we write $\Delta_H(u)$ for $\Delta_{V(H)}(u)$. Let P be the matrix of the orthogonal projection of \mathbb{R}^n onto $\mathcal{E}(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of \mathbb{R}^n .

We shall require the following properties of star sets and star complements; the first follows from [7, Proposition 5.1.1].

Lemma 1.1 *The subset S of $V(G)$ lies in a star set for μ if and only if the vectors $P\mathbf{e}_i$ ($i \in S$) are linearly independent.*

Since P is a polynomial in A [7, Equation 1.5] we have $\mu P\mathbf{e}_i = AP\mathbf{e}_i = PA\mathbf{e}_i$ ($i = 1, \dots, n$), whence:

Lemma 1.2 $\mu P\mathbf{e}_i = \sum_{j \sim i} P\mathbf{e}_j$ ($i = 1, \dots, n$).

As a consequence of Lemmas 1.1 and 1.2 we have:

Lemma 1.3 [7, Proposition 5.1.4] *Let X be a star set for μ in G , and let $H = G - X$. If $\mu \notin \{-1, 0\}$, then $V(H)$ is a location-dominating set in G , that is, the H -neighbourhoods $\Delta_H(u)$ ($u \in X$) are non-empty and distinct.*

By interlacing [7, Corollary 1.3.12] we have:

Lemma 1.4 *If S is a star set for μ in G and if U is a proper subset of S then $S \setminus U$ is a star set for μ in $G - U$.*

The next result strengthens [1, Theorem 2.3], which says that if G has H as a star complement of order t , for an eigenvalue $\mu \notin \{-1, 0\}$, then either (a) G has order at most $\binom{t+1}{2}$, or (b) $\mu = 1$ and $G = K_2$ or $2K_2$.

Proposition 1.5 *Let G be a graph with X as a star set for μ , and let $H = G - X$. Let $s = |\cup_{i \in X} \Delta_H(i)|$.*

- (i) *If $|X| > s$ then G_X has μ as an eigenvalue of multiplicity at least $|X| - s$.*
- (ii) *If $\mu \notin \{-1, 0\}$ then $|X| \leq \binom{s+1}{2}$.*

Proof. Since $s \leq \binom{s+1}{2}$, the second assertion is immediate when $|X| \leq s$. Accordingly we assume throughout the proof that $|X| > s$. We show first that μ is an eigenvalue of G_X . Let $S = \cup_{i \in X} \Delta_H(i)$ and $X = \{1, 2, \dots, k\}$. By Lemma 1.2, the vectors $\mu P\mathbf{e}_i - \sum\{P\mathbf{e}_j : j \in \Delta_X(i)\}$ ($i \in X$) lie in the subspace $\langle P\mathbf{e}_h : h \in S \rangle$, and so there exist $\alpha_1, \alpha_2, \dots, \alpha_k$, not all zero, such that

$$\sum_{i=1}^k \alpha_i (\mu P\mathbf{e}_i - \sum\{P\mathbf{e}_j : j \in \Delta_X(i)\}) = \mathbf{0}.$$

Since the vectors $P\mathbf{e}_i$ ($i \in X$) are linearly independent, it follows that $(\mu I - A_X)\mathbf{a} = \mathbf{0}$, where $\mathbf{a} = (\alpha_1, \alpha_2, \dots, \alpha_k)^\top$ and A_X is the adjacency matrix of G_X .

Let Y be a star set for μ in G_X and consider the graph $G - Y$. If $|X| - |Y| > s$ then the above argument shows that μ is an eigenvalue of $G_{X \setminus Y}$. This is a contradiction because $G_{X \setminus Y}$ is a star complement for μ in G_X . Hence $|Y| \geq |X| - s$, and we have proved the first assertion.

Since G_X has a star complement $G_{X \setminus Y}$ of order at most s , we have $|X| \leq \binom{s+1}{2}$ whenever $\mu \notin \{-1, 0\}$. (Note that by Lemma 1.3, we have $G_X \neq K_2$, while $s = 3$ when $G_X = 2K_2$.) \square

2 Controlling connectivity

Let G be a graph with a k -connected star complement H for an eigenvalue other than -1 or 0 . In effect the following result establishes a quadratic upper bound for $\delta(G)$ in the case that G is not k -connected. The vertex-connectivity of G is denoted by $\kappa(G)$, and we refer to a separating set of size $\kappa(G)$ as a *minimum* separating set.

Theorem 2.1 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . If $\kappa(H) \geq k \geq 1$ and $\delta(G) \geq \frac{1}{2}(k-1)(k+2)$ then $\kappa(G) \geq k$.*

Proof. The proof is by induction on k ; the result holds for $k = 1$ since $V(H)$ is a dominating set in G . Assume that $k > 1$ and that the result holds for $k-1$. Suppose by way of contradiction that $\kappa(G) < k$; then $\kappa(G) = k-1$ by the induction hypothesis.

Let S be a cutset in G of size $k-1$. If S contains a vertex v outside H then (by Lemma 1.4) $G - v$ has H as a star complement, while $\delta(G - v) \geq \frac{1}{2}(k-1)(k+2) - 1 > \frac{1}{2}(k-2)(k+1)$. By the induction hypothesis, we have $\kappa(G - v) \geq k-1$. This is a contradiction because $G - v$ has $S \setminus \{v\}$ as a separating set of size $k-2$.

Thus $S \subseteq V(H)$. Since $H - S$ is connected, $H - S$ lies in some component C of $G - S$. Let $R = V(G - S) \setminus V(C)$. Note that if $j \in R$ then $\Delta_H(j) \subseteq S$, and so $|\Delta_R(j)| \geq \frac{1}{2}(k-1)(k+2) - (k-1) = \frac{1}{2}k(k-1)$. Hence $|R| > \binom{k}{2}$.

However, $|\cup_{j \in R} \Delta_H(j)| \leq k - 1$, and so by Proposition 1.5(ii) we have $|R| \leq \binom{k}{2}$, a contradiction. \square

Corollary 2.2 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . Then $\kappa(G)$ is controlled by $\kappa(H)$ and $\delta(G)$.*

Proof. Let $\kappa = \kappa(H)$, $\delta = \delta(G)$, and $F(k) = \frac{1}{2}(k-1)(k+2)$ ($k \in \mathbb{N}$). Define $g(\kappa, \delta)$ as the largest $k \leq \kappa$ such that $F(k) \leq \delta$. Then $\kappa(G) \geq g(\kappa, \delta)$ by the Theorem. Since also $\kappa(G) \leq \delta$, the result follows. \square

In view of Theorem 2.1, we may define $f(k)$ ($k \in \mathbb{N}$) as the least non-negative integer such that $\kappa(G) \geq k$ whenever $\kappa(H) \geq k$ and $\delta(G) \geq f(k)$. Thus $f(k) \leq F(k)$ for all ($k \in \mathbb{N}$). The following example shows that $f(k) \geq k$ for all $k > 1$.

Example 2.3 For $k \geq 2$, let G_k be the graph obtained from a $(k+1)$ -clique H_k by adding a vertex of degree $k-1$, and let μ be the largest eigenvalue of G_k . Since G_k is connected, we have $\mu > k$ and so H_k is a star complement for μ . Now $\kappa(G_k) = k-1 = \delta(G_k)$, while $\kappa(H_k) = k$. Hence $f(k) \geq k$. \square

Since $F(2) = 2$, we have $f(2) = 2$, an observation which follows also from [8, Proposition 2.1(ii)]. We investigate $f(k)$ ($k = 3, 4, 5, 6$) in the next two sections; there we shall require the following result, which is proved by refining an argument in the proof of Theorem 2.1.

Proposition 2.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . If $\kappa(H) \geq k > 1$, $\delta(G) > f(k-1)$ and $\kappa(G) < k$ then every minimum separating set for G lies in $V(H)$.*

Proof. Let S be a minimum separating set for G , and suppose by way of contradiction that S contains a vertex v outside H . Then H is a star complement for μ in $G - v$. Since $\kappa(H) \geq k-1$ and $\delta(G-v) \geq f(k-1)$, we have $\kappa(G-v) \geq k-1$. This is a contradiction because $G-v$ has $S \setminus \{v\}$ as a separating set of size at most $k-2$. \square

3 The cases $\kappa(H) = 3, 4$

In this section we determine $f(3)$ and $f(4)$. We make use of the following observation.

Lemma 3.1 *Let G be a graph with X as a star set for an eigenvalue other than -1 or 0 , and let $H = G - X$. Then X does not contain vertices $1, 2, 3$ such that $\Delta_H(1)$ is the disjoint union of $\Delta_H(2)$ and $\Delta_H(3)$.*

Proof. Suppose by way of contradiction that $\Delta_H(1) = \Delta_H(2) \dot{\cup} \Delta_H(3)$. By Lemma 1.4, we may take $X = \{1, 2, 3\}$. Let $|V(G)| = n$ and let P be the orthogonal projection of \mathbb{R}^n onto $\mathcal{E}(\mu)$, so that

$$\Sigma\{P\mathbf{e}_i : i \in \Delta_H(1)\} = \Sigma\{P\mathbf{e}_j : j \in \Delta_H(2)\} + \Sigma\{P\mathbf{e}_k : k \in \Delta_H(3)\}.$$

By Lemma 1.2, we have

$$\begin{aligned} \mu P\mathbf{e}_1 - \Sigma\{P\mathbf{e}_i : i \in \Delta_X(1)\} &= \Sigma\{P\mathbf{e}_i : i \in \Delta_H(1)\} = \\ \mu P\mathbf{e}_2 - \Sigma\{P\mathbf{e}_j : j \in \Delta_X(2)\} + \mu P\mathbf{e}_3 - \Sigma\{P\mathbf{e}_k : k \in \Delta_X(3)\}. \end{aligned} \quad (1)$$

We examine the various possibilities for G_X . To within a transposition of the vertices 2 and 3, there are six cases to consider, namely those in which the edge-set of G_X is one of $\{12, 13, 23\}$, $\{12, 23\}$, $\{12, 13\}$, $\{12\}$, $\{23\}$, \emptyset . Equation (1) becomes respectively:

$$\begin{aligned}\mu Pe_1 - Pe_2 - Pe_3 &= \mu Pe_2 - Pe_1 - Pe_3 + \mu Pe_3 - Pe_1 - Pe_2, \\ \mu Pe_1 - Pe_2 &= \mu Pe_2 - Pe_1 - Pe_3 + \mu Pe_3 - Pe_2, \\ \mu Pe_1 - Pe_2 - Pe_3 &= \mu Pe_2 - Pe_1 + \mu Pe_3 - Pe_1, \\ \mu Pe_1 - Pe_2 &= \mu Pe_2 - Pe_1 + \mu Pe_3, \\ \mu Pe_1 &= \mu Pe_2 - Pe_3 + \mu Pe_3 - Pe_2, \\ \mu Pe_1 &= \mu Pe_2 + \mu Pe_3.\end{aligned}$$

By Lemma 1.1, the vectors Pe_1, Pe_2, Pe_3 are linearly independent, and this leads to a contradiction in all cases. \square

Proposition 3.2 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . If H is 3-connected and $\delta(G) \geq 3$ then G is 3-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 3$. Since $f(2) = 2$, we have $\kappa(G) = 2$. Let $H = G - X$ and let S be a minimum separating set in G , say $S = \{v, w\}$. Then $S \subseteq V(H)$ by Proposition 2.4. As in the proof of Theorem 2.1, let R be the set of vertices in X that lie outside the component of $G - S$ containing $H - S$. Then $\Delta_H(u) \subseteq S$ for all $u \in R$. Since $\delta(G) \geq 3$, while the neighbourhoods $\Delta_H(u)$ ($u \in R$) are non-empty and distinct, R consists of 3 pairwise adjacent vertices whose H -neighbourhoods are $\{v\}, \{w\}$ and $\{v, w\}$. Now Lemma 3.1 provides a contradiction. \square

It follows that $f(3) = 3$. We begin our investigation of $f(4)$ with an example which demonstrates that $f(4) > 4$. (This example was found experimentally using the computer package GRAPH [4].)

Example 3.3 (Here non-integer eigenvalues are given to 4 decimal places.) Let H be the graph obtained from an 8-clique by deleting an edge, and let u, v, w be pairwise adjacent vertices in H . Let G be the graph obtained from H by adding three pairwise adjacent vertices with H -neighbourhoods $\{u, v\}, \{u, w\}, \{v, w\}$. The spectrum of G is 7.1017, 2.3416, $0^{(3)}, -1^{(2)}, -1.4433, -2^{(3)}$, and the spectrum of H is 6.7720, $0, -1^{(5)}, -1.7720$. Thus G has H as a star complement for -2 . Now $\kappa(H) = 6$ and $\delta(G) = 4$, while $\kappa(G) = 3$. It follows that $f(4) \geq 5$. \square

Proposition 3.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . If H is 4-connected and $\delta(G) \geq 5$ then G is 4-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 4$. Since $f(3) = 3$, we have $\kappa(G) = 3$. Let $H = G - X$ and let S be a minimum separating set in G , say $S = \{u, v, w\}$. Then $S \subseteq V(H)$ by Proposition 2.4. Defining R as before, we have $|R| \leq 7$ by Lemma 1.3. On the other hand, $|R| \geq 4$ because $\delta(G) \geq 5$. Moreover, if $|R| = 4$ then R induces a clique and the H -neighbourhoods of the vertices in R are the four subsets of size 2 or 3.

In this case, let $R = \{1, 2, 3, 4\}$, with $\Delta_H(1) = \{u, v, w\}$, $\Delta_H(2) = \{v, w\}$, $\Delta_H(3) = \{u, w\}$ and $\Delta_H(4) = \{u, v\}$. By Lemma 1.4, we may take $X = R$. Defining P as before, we have from Lemma 1.2:

$$2(\mu P\mathbf{e}_1 - \Sigma\{P\mathbf{e}_i : i \in \Delta_X(1)\}) = 2\Sigma\{P\mathbf{e}_i : i \in S\} = \sum_{j=2}^4 (\mu P\mathbf{e}_j - \Sigma\{P\mathbf{e}_k : k \in \Delta_X(j)\}).$$

It follows that $(2\mu + 3)P\mathbf{e}_1 = \mu(P\mathbf{e}_2 + P\mathbf{e}_3 + P\mathbf{e}_4)$, and hence that $P\mathbf{e}_1, P\mathbf{e}_2, P\mathbf{e}_3, P\mathbf{e}_4$ are linearly dependent, contradicting Lemma 1.1.

We deduce that $|R| \in \{5, 6, 7\}$. Now the neighbourhoods $\Delta_H(i)$ ($i \in R$) must include two disjoint subsets whose union is a third such neighbourhood, and Lemma 3.1 provides a final contradiction. \square

We deduce that $f(4) = 5$.

4 The cases $\kappa(H) = 5, 6$

We begin with two examples which show that $f(5) \geq 7$ and $f(6) \geq 8$. Here we make use of the following result from [10, Section 3] (see also [6, Example 5.1.14] and [7, Example 5.2.16]).

Proposition 4.1 *Let G be a finite graph with a proper induced subgraph $G - X = H \cong K_8$ such that*

- (i) $|\Delta_H(u)| = 3$ for all $u \in X$,
- (ii) for distinct vertices u, v of X , $|\Delta_H(u) \cap \Delta_H(v)| = 2$ when $u \sim v$, and $|\Delta_H(u) \cap \Delta_H(v)| = 1$ when $u \not\sim v$.

Then H is a star complement in G for the eigenvalue -2 .

Example 4.2 Let G be the graph obtained from an 8-clique by adding four pairwise adjacent vertices whose H -neighbourhoods are the four 3-subsets of a 4-set in $V(H)$. By Proposition 4.1, G has H as a star complement for -2 . Now $\kappa(H) = 7$, $\delta(G) = 6$ and $\kappa(G) = 4$. Hence $f(5) > 6$. \square

Example 4.3 Here G is constructed from an 8-clique H by first adding a set X of ten vertices whose H -neighbourhoods are the ten 3-subsets of a 5-set in $V(H)$; secondly, we add an edge between vertices u and v of X whenever $|\Delta_H(u) \cap \Delta_H(v)| = 2$. (It follows that X induces the line graph $L(K_5)$.) By Proposition 4.1, G has H as a star complement for -2 . We have $\kappa(H) = 7$, $\delta(G) = 7 = \delta(H)$ and $\kappa(G) = 5$. Hence $f(6) > 7$. \square

Since $F(5) = 14$ and $F(6) = 20$, we deduce that $7 \leq f(5) \leq 14$ and $8 \leq f(6) \leq 20$. We conclude by showing that $f(5) = 7$.

Proposition 4.4 *Let G be a graph with H as a star complement for an eigenvalue other than -1 or 0 . If H is 5-connected and $\delta(G) \geq 7$ then G is 5-connected.*

Proof. Suppose by way of contradiction that $\kappa(G) < 5$. Since $f(4) = 5$, we have $\kappa(G) = 4$. Let $H = G - X$ and let S be a minimum separating set in G . Then $S \subseteq V(H)$ by Proposition 2.4. Defining R as before, we have $|R| \geq 5$ because $\delta(G) \geq 7$. Moreover, if $|R| = 5$ then R induces a clique and

the H -neighbourhoods of the vertices in R are the five subsets of size 3 or 4. If $R = \{1, 2, 3, 4, 5\}$, with $\Delta_H(1) = S$ then (arguing as before) we find that $(3\mu + 4)Pe_1 = \mu(Pe_2 + Pe_3 + Pe_4 + Pe_5)$, contradicting Lemma 1.1.

Next suppose that $|R| = 6$. Applying Proposition 1.5(i) with $s \leq 4$ we see that μ is a multiple eigenvalue of G_R . Now each vertex of G_R has degree at least 3, with at most one vertex of degree equal to 3, and so each vertex of the complement $\overline{G_R}$ has degree at most 2, with at most one vertex of degree equal to 2. Thus G_R is one of the graphs numbered 1,2,3,4,8,9 in [5]. Only the last of these has a multiple eigenvalue other than -1 or 0 , and it follows that G_R is the skeleton of an octahedron (with $\mu = -2$). Hence $|\Delta_H(j)| \geq 3$ for all $j \in R$. This contradicts Lemma 1.3 because S has only five distinct subsets of size at least 3.

Finally, suppose that $|R| \geq 7$. Let Q be a 7-subset of R and consider the subgraph G' of G induced by $V(H) \cup Q$. By Proposition 1.5(i), $G_Q (= G'_Q)$ has μ as an eigenvalue of multiplicity at least 3. Hence $G_Q = 3K_2 \cup K_1$, with $\mu = 1$. (This follows from an inspection of the spectra of (i) the connected graphs of order 7 [2, pp.176-232] and (ii) the connected graphs of order at most 6 [3, 5]. Note that in case (i), if μ is not an integer then μ has an algebraic conjugate $\mu^* \neq \mu$ with the same multiplicity. In this situation, $\mu + \mu^*$ is an integer and the largest eigenvalue λ_1 of G is $-3(\mu + \mu^*)$; since G_Q is not complete, necessarily $\lambda_1 = 3$.)

Let $Q = \{1, 2, 3, 4, 5, 6, 7\}$, with $1 \sim 2$, $3 \sim 4$, $5 \sim 6$, and let $W = \langle Pe_h : h \in S \rangle$, where now P represents the orthogonal projection onto $\mathcal{E}(\mu)$ associated with G' . The subspace W has dimension 4 because it contains the vectors $Pe_1 - Pe_2$, $Pe_3 - Pe_4$, $Pe_5 - Pe_6$ and Pe_7 . Hence the vectors Pe_h ($h \in S$) are linearly independent. Now

$$Pe_1 - Pe_2 = \Sigma\{Pe_i : i \in \Delta_S(1)\}, \quad Pe_2 - Pe_1 = \Sigma\{Pe_i : i \in \Delta_S(2)\}.$$

It follows that

$$\begin{aligned} & \Sigma\{Pe_h : h \in \Delta_S(1) \setminus \Delta_S(2)\} + \Sigma\{Pe_h : h \in \Delta_S(2) \setminus \Delta_S(1)\} \\ & \quad + 2\Sigma\{Pe_h : h \in \Delta_S(1) \cap \Delta_S(2)\} = \mathbf{0}, \end{aligned}$$

a final contradiction. \square

Combining the various results from Sections 2, 3 and 4, we have:

Theorem 4.5 *Let G be a graph with H as a star complement for the eigenvalue μ . For each $k \in \mathbb{N}$ there exists a smallest non-negative integer $f(k)$ with the property*

$$\mu \notin \{-1, 0\}, \quad \kappa(H) \geq k, \quad \delta(G) \geq f(k) \Rightarrow \kappa(G) \geq k.$$

We have $k \leq f(k) \leq \frac{1}{2}(k-1)(k+2)$ for all $k \geq 2$; moreover, $f(1) = 0$, $f(2) = 2$, $f(3) = 3$, $f(4) = 5$, $f(5) = 7$ and $f(6) \geq 8$. (In particular, if $\mu \notin \{-1, 0\}$ and H is connected, then G is connected, while if $\mu \notin \{-1, 0\}$ and H is 2-connected then either G is 2-connected or G has a pendant vertex.)

Acknowledgment. The author is grateful to two anonymous referees for their constructive comments.

References

- [1] F. K. Bell and P. Rowlinson, On the multiplicities of graph eigenvalues, *Bull. London Math. Soc.* 35 (2003), 401-408.
- [2] D. Cvetković, M. Doob, I. Gutman and A. Torgašev. *Recent Results in the Theory of Graph Spectra*, North-Holland (Amsterdam), 1988.
- [3] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, 3rd edn., Johann Ambrosius Barth (Heidelberg), 1996.
- [4] D. Cvetković, L. Kraus and S. K. Simić, Discussing graph theory with a computer I: implementation of graph-theoretic algorithms, *Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz.*, Nos. 716-734 (1981), 100-104.
- [5] D. Cvetković and M. Petrić, A table of connected graphs on six vertices, *Discrete Math* 50 (1984), 37-49.
- [6] D. Cvetković, P. Rowlinson and S. K. Simić, *Spectral Generalizations of Line Graphs*, Cambridge University Press (Cambridge), 2004.
- [7] D. Cvetković, P. Rowlinson and S. K. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press (Cambridge), 2010.
- [8] P. Rowlinson, Star sets and star complements in finite graphs: a spectral construction technique, in: *Discrete Mathematical Chemistry*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol.51 (eds. P. Hansen, P. Fowler and M. Zheng), American Math. Soc. (Providence, RI) 2000, pp.323-332.
- [9] P. Rowlinson, Star complements in finite graphs: a survey, *Rendiconti Sem. Mat. Messina* 8 (2002), 145-162.
- [10] P. Rowlinson, Star complements and maximal exceptional graphs, *Publ. Inst. Math. Beograd* 76(90) (2004), 25-30.
- [11] P. Rowlinson and B. Tayfeh-Rezaie, Star complements in regular graphs: old and new results, *Linear Algebra Appl.* 432 (2010), 2230-2242.